Speed of convergence towards attracting sets for endomorphisms of $\mathbb{P}^k$

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Abstract

Let f be a non-invertible holomorphic endomorphism of $\mathbb{P}^k$ having an attracting set A. We show that, under some natural assumptions, A supports a unique invariant positive closed current τ, of the right bidegree and of mass 1. Moreover, if R is a current supported in a small neighborhood of A then its push-forwards by f^n converge to τ exponentially fast. We also prove that the equilibrium measure on A is hyperbolic.

1 Introduction

Let f be a holomorphic endomorphism of algebraic degree $d \geq 2$ on the complex projective space $\mathbb{P}^k$. A compact subset A of $\mathbb{P}^k$ is called an *attracting set* if it has a *trapping neighborhood* U i.e. $f(U) \Subset U$ and $A = \cap_{n\geq 0} f^n(U)$ where $f^n := f \circ \cdots \circ f$, n times. It follows that A is invariant, $f(A) = A$. Furthermore, if A contains a dense orbit then A is a *trapped attractor*. Typical examples of such objects are fractal and their underlying dynamics are hard to study. We refer to [Mil85], [Rue89] for general discussions on attractors and to [FW99], [JW00], [FS01], [BDM07], [Taf10] and references therein for examples of different types of attractors in $\mathbb{P}^2$.

Attracting sets are stable under small perturbations. Indeed, if f has an attracting set $A = \cap_{n\geq 0} f^n(U)$ then any small perturbation f_ϵ of f has an attracting set defined by $A_\epsilon = \cap_{n\geq 0} f_\epsilon^n(U)$. For example, when f restricted to $\mathbb{C}^k$ defines a polynomial self-map then the hyperplane at infinity $\mathbb{P}^k \setminus \mathbb{C}^k$ is an attracting set. In the same way, it is easy to create examples where the attracting set is a projective subspace of arbitrary dimension. In this paper, we consider a family of endomorphisms, stable under small perturbations, which contains these examples. It was introduced by Dinh in [Din07] and we briefly recall the context.

In the sequel, we always assume that f possesses an attracting set A which has a trapped neighborhood U satisfying the following properties. There exist an integer $1 \leq p \leq k-1$ and two projective subspaces I and L of dimension $p-1$ and $k-p$ respectively such that $I \cap U = \varnothing$ and $L \subset U$. We do not assume that L and I are invariant. Since $I \cap L = \varnothing$, for each $x \in L$ there exists a unique projective subspace $I(x)$ of dimension p which contains I and such that $L \cap I(x) = \{x\}$. Furthermore, for each $x \in L$ we ask that $U \cap I(x)$ is strictly convex as a subset of $I(x) \setminus I \simeq \mathbb{C}^p$. All these assumptions are stable under small perturbations of f. The geometric assumption on U is slightly stronger than the one of Dinh, who only requires $U \cap I(x)$ to be star-shaped at x. We need convexity in order the solve the $\overline{\partial}$-equation on U. Indeed, under our assumption U is a $(p-1)$-convex domain in the sense of [HL88].

If E is a subset of $\mathbb{P}^k$, let $\mathscr{C}_q(E)$ denote the set of all positive closed currents of bidegree (q,q), supported in E and of mass 1. It is well known that for any integer $1 \leq q \leq k$ and any smooth form S in $\mathscr{C}_q(\mathbb{P}^k)$, the sequence $d^{-qn}(f^n)^*(S)$ converges to a positive closed current T^q of bidegree (q,q) called the *Green current of order* q of f. We refer to [DS10] for a detailed exposition on these currents and their effectiveness in holomorphic dynamics.

When $q = k$, it gives the *equilibrium measure* of f, $\mu := T^k$. It is exponentially mixing and it is the unique measure of maximal entropy $k \log d$ on $\mathbb{P}^k$. Moreover, it is hyperbolic and all its Lyapunov exponents are larger or equal to $(\log d)/2$. The dynamics outside the support of μ is not very well understood. The aim of this paper is to continue the investigation started in [Din07] on the attracting sets described above, which do not intersect $\mathrm{supp}(\mu)$. Indeed, since $I \cap U = \varnothing$, by regularization there exists a smooth form $S \in \mathscr{C}_{k-p+1}(\Omega)$, where $\Omega := \mathbb{P}^k \setminus \overline{U}$. As $f^{-1}(\Omega) \subset \Omega$, it follows that $\mathrm{supp}(T^{k-p+1}) \cap U = \varnothing$, and hence $\mathrm{supp}(T^q) \cap U = \varnothing$ if $q \geq k-p+1$.

The set $\mathscr{C}_p(U)$ is non-empty since it contains the current $[L]$ of integration on L and its regularizations in U. In the situation described above, Dinh proved that if R is a continuous element of $\mathscr{C}_p(U)$ then its normalized push-forwards by f^n, $d^{-(k-p)n}(f^n)_*(R)$, converge to a current τ which is independent of the choice of R. Moreover, the current τ gives us information on the geometry of A and on the dynamics of $f_{|A}$: it is woven, supported in A and invariant i.e. $f_*(\tau) = d^{k-p}\tau$. Our main result is that, with a natural additional assumption on $f_{|U}$, stable under small perturbations, we obtain an explicit exponential speed of the above convergence for any R in $\mathscr{C}_p(U)$.

Theorem 1.1. *Let f and τ be as above and assume that $\|\wedge^{k-p+1} Df(z)\| < 1$ on $\overline{U}$. There is a constant $0 < \lambda < 1$ such that for each $0 < \alpha \leq 2$ the* *following property holds. There exists $C > 0$ such that for any element R of $\mathscr{C}_p(U)$ and any $(k-p, k-p)$-form φ of class $\mathcal{C}^\alpha$ on $\mathbb{P}^k$ we have*

$$|\langle d^{-(k-p)n}(f^n)_*(R) - \tau, \varphi\rangle| \leq C\lambda^{n\alpha/2}\|\varphi\|_{\mathcal{C}^\alpha}. \tag{1.1}$$

In particular, τ is the unique invariant current in $\mathscr{C}_p(U)$ and $d^{-(k-p)n}(f^n)_*(R)$ converge to τ uniformly on $R \in \mathscr{C}_p(U)$.

Recall that f induces a self-map Df on the tangent bundle $T\mathbb{P}^k$ which also gives a self-map $\wedge^q Df$ on the exterior power $\wedge^q T\mathbb{P}^k$, $1 \leq q \leq k$. In the sequel, all the norms on $\mathcal{C}^\alpha$, L^r, etc. are with respect to the Fubini-Study metric on $\mathbb{P}^k$. It also gives a uniform norm which induces an operator norm for $\wedge^q Df$.

In the same spirit as Theorem 1.1, we proved in [Taf11] that for a generic current S in $\mathscr{C}_1(\mathbb{P}^k)$, the sequence $d^{-n}(f^n)^*(S)$ converges to T exponentially fast. However, the contexts are quite different. Here, we consider currents of arbitrary bidegree which are in general much harder to handle. Moreover, in [Taf11] we deeply use that T has Hölder continuous local potentials. In the present situation, we can expect that the attracting current τ is always more singular. The idea to prove Theorem 1.1 is to use Henkin-Leiterer's solution with estimates of the dd^c-equation on U in order to study separately the harmonic and non-harmonic parts of the left hand side term of (1.1). When $dd^c\varphi = 0$ on U, we use the "geometry" of $\mathscr{C}_p(U)$, introduced in [Din07] and [DS06], and Harnack's inequality to obtain exponential estimates. In order to deal with the non-harmonic part, we use the assumption on $\| \wedge^{k-p+1} Df\|$. This assumption comes naturally in several basic examples and their perturbations.

In [Din07], Dinh also showed that the *equilibrium measure associated to* A, defined by $\nu := \tau \wedge T^{k-p}$, is invariant, mixing and of maximal entropy $(k-p)\log d$ on A. Theorem 1.1 is a first step in order to obtain other ergodic and stochastic properties on ν as exponential mixing or central limit theorem. We postpone this question in a future work.

Under the same assumptions, we deduce from the work of de Thélin [dT08], see also [Dup09], the following result on ν.

Theorem 1.2. *If f is as in Theorem 1.1, then the measure ν is hyperbolic. More precisely, counting with multiplicity it has $k-p$ Lyapunov exponents larger than or equal to $(\log d)/2$ and p Lyapunov exponents strictly smaller than $-(k-p)(\log d)/2$.*

2 Structural discs of currents

In this section we recall the notion of structural varieties of currents. It was introduced by Dinh and Sibony in order to put a geometry on the space $\mathscr{C}_p(U)$ which is of infinite dimension, see [DS06] and [Din07]. The definition of structural varieties is based on slicing theory and they can be seen as complex subvarieties inside $\mathscr{C}_p(U)$. In [DS09], the authors developed the notion of super-potential which involves more deeply this geometry.

Slicing theory can be seen as a generalization to currents of restriction of smooth forms to submanifolds. We will briefly explain it in our context and refer to [Fed69] for a more complete account.

Let U be an open subset of $\mathbb{P}^k$ satisfying the geometric hypothesis as above. Let V be a complex manifold of dimension l. We denote by π_U and π_V the canonical projections of $U \times V$ to U and V respectively. If $\mathscr{R}$ is a positive closed current of bidegree (p,p) on $U \times V$ with $\pi_U(\text{supp}(\mathscr{R})) \Subset U$ then, for all θ in V, the slice $\langle \mathscr{R}, \pi_V, \theta \rangle$ is well defined. For any $(k-p, k-p)$-form ϕ on $U \times V$ we have

$$\langle \mathscr{R}, \pi_V, \theta \rangle(\phi) = \lim_{\epsilon \to 0} \langle \mathscr{R} \wedge \pi_V^*(\psi_{\theta,\epsilon}), \phi \rangle,$$

where $\psi_{\theta,\epsilon}$ is an appropriate approximation in V of the Dirac mass at θ. It is a $(p+l, p+l)$-current on $U \times V$ supported on $U \times \{\theta\}$ which can be identified to a (p,p)-current on U. A family of currents $(R_\theta)_{\theta \in V}$ in $\mathscr{C}_p(U)$ is a *structural variety* if there exists a positive closed current $\mathscr{R}$ in $U \times V$ such that $R_\theta = \langle \mathscr{R}, \pi_V, \theta \rangle$. When V is isomorphic to the unit disc of $\mathbb{C}$, we call $(R_\theta)_{\theta \in V}$ a *structural disc.*

Recall that in our situation $f(U) \Subset U$. Under the geometrical assumption on U, Dinh constructed in [Din07, p.233] a family of structural discs in $\mathscr{C}_p(U)$. He uses that for each $x \in L$ the set $I(x) \cap U$ is star-shaped at x to obtain a property similar to star-sharpness for $\mathscr{C}_p(U)$.

More precisely, up to an automorphism, we can assume that

$$I = \{x \in \mathbb{P}^k \,|\, x_i = 0,\ 0 \leq i \leq k-p\},\ L = \{x \in \mathbb{P}^k \,|\, x_i = 0,\ k-p+1 \leq i \leq k\},$$

where $x = [x_0 : \cdots : x_k]$ are the homogeneous coordinates of $\mathbb{P}^k$. For $\theta \in \mathbb{C}$, $A_\theta(x) := [x_0 : \cdots : x_{k-p} : \theta x_{k-p+1} : \cdots : \theta x_k]$ is an automorphism of $\mathbb{P}^k$ except for $\theta = 0$ where it is the projection of $\mathbb{P}^k \setminus I$ on L. Let set $U' := f(U)$. As $I(x) \cap U$ is star-shaped at $x \in L$, there exists a simply connected open neighborhood $V \subset \mathbb{C}$ of $[0,1]$ such that $A_\theta(U') \Subset U$ for all θ in $\overline{V}$. If S is in $\mathscr{C}_p(U')$ then the family $(S_\theta)_{\theta \in V}$ with $S_\theta := (A_\theta)_*(S)$ defined a structural disc. Indeed, if $\Lambda : \mathbb{P}^k \times V \to \mathbb{P}^k$ is the meromorphic map defined by $\Lambda(x,\theta) = (A_\theta)^{-1}(x)$ and $\mathscr{S} := \Lambda^* S$ then $S_\theta = \langle \mathscr{S}, \pi_V, \theta \rangle$, see [Din07] for more details. For any S in $\mathscr{C}_p(U')$, we have that $S_1 = S$ and $S_0 = [L]$ which is independent of S. In other words, any current in $\mathscr{C}_p(U')$ is linked to $[L]$ by a structural disc in $\mathscr{C}_p(U)$. Moreover, S_θ depends continuously on θ and we have the following important property.

Lemma 2.1 ([Din07]). *Let S be in $\mathscr{C}_p(U')$ and $(S_\theta)_{\theta\in V}$ be the structural disc described above. If ϕ is a real continuous $(k-p,k-p)$-form with $dd^c\phi = 0$ on U then the function $\theta \mapsto \langle S_\theta, \phi\rangle$ is harmonic on V.*

3 q-Convex set and d-bar equation

The concept of q-convexity generalizes both Stein and compact manifolds. Andreotti and Grauert [AG62] obtained vanishing or finiteness theorems for q-convex manifolds and, in [HL88], Henkin and Leiterer developed a similar theory using integral representations. In particular, they obtained solutions of the $\overline{\partial}$-equation with explicit estimates, which play a key role in our proof. For this reason, we use the conventions of [HL88].

If $1 \leq q \leq k$ is an integer then a real $\mathcal{C}^2$ function ρ on an open subset $V \subset \mathbb{P}^k$ is called *q-convex* if, in any holomorphic local coordinates, the Hermitian form

$$L_\rho(x)t = \sum_{i,j=1}^{k} \frac{\partial^2\rho(x)}{\partial z_i \partial \overline{z}_j} t_i \bar{t}_j$$

has at least q strictly positive eigenvalues at any point $x \in V$.

Let $0 \leq q \leq k-1$. We say that an open subset D of $\mathbb{P}^k$ is *strictly q-convex* if there exists a $(q+1)$-convex function ρ in a neighborhood V of ∂D such that

$$D \cap V = \{x \in V \mid \rho(x) < 0\}.$$

Moreover, if the same condition is satisfied with V a neighborhood of $\overline{D}$ then D is called *completely strictly q-convex.*

The strict q-convexity has the following important consequence, see [HL88, Theorem 11.2].

Theorem 3.1. *Let D be a strictly q-convex open subset of $\mathbb{P}^k$ with $\mathcal{C}^2$ boundary. If ϕ is a continuous $\overline{\partial}$-exact form of bidegree (r,s) in a neighborhood of $\overline{D}$ with $0 \leq s \leq k$, $k-q \leq r \leq k$, then there exists a continuous $(s, r-1)$-form ψ on D such that $\overline{\partial}\psi = \phi$ and*

$$\|\psi\|_{\infty,D} \leq C\|\phi\|_{\infty,D}$$

for some $C > 0$ independent of ϕ.

Furthermore, Andreotti and Grauert proved the following vanishing theorem, see [AG62] and [HL88, Theorem 12.7].

Theorem 3.2. *If D is a completely strictly q-convex open subset of $\mathbb{P}^k$ with $\mathcal{C}^2$ boundary then $H^{s,r}(D, \mathbb{C}) = 0$ for any $0 \leq s \leq k$ and $k - q \leq r \leq k$.*

Henkin and Leiterer [HL88, Theorem 5.13] give the following criteria of q-convexity, which is closely related to our geometric assumption on U with $q = p - 1$.

Theorem 3.3. *Let D be an open subset of $\mathbb{P}^k$ with $\mathcal{C}^2$ boundary. If for each $x \in \partial D$ there exists a complex submanifold $Y \subset \mathbb{P}^k$ of dimension $q + 1$, transverse to ∂D and such that $Y \cap D$ is a strictly pseudoconvex domain in Y, then D is strictly q-convex.*

This result applies to our trapping neighborhood U with $q = p-1$. Indeed, observe that, possibly by exchanging U by a slightly smaller open set which contains $f(U)$, we can assume that ∂U is smooth and the intersection of ∂U with $I(x)$ is transverse for all $x \in L$. The projective space $I(x)$ has dimension $p = q + 1$ and $U \cap I(x)$ is strictly convex in $I(x) \setminus I \simeq \mathbb{C}^p$, so in particular strictly pseudoconvex in $I(x)$. Therefore, by Theorem 3.3, U is strictly $(p - 1)$-convex. In the sequel, we always choose such an attracting neighborhood U.

Up to an automorphism of $\mathbb{P}^k$, I is defined in homogeneous coordinates by $I = \{x \in \mathbb{P}^k \,|\, x_i = 0,\ 0 \leq i \leq k - p\}$. The function

$$\eta(x) = \frac{|x_{k-p+1}|^2 + \cdots + |x_k|^2}{|x_0|^2 + \cdots + |x_{k-p}|^2},$$

is a $(q + 1)$-convex exhausting function of $\mathbb{P}^k \setminus I$, i.e. $\mathbb{P}^k \setminus I$ is *completely q-convex.* In general, strictly q-convex subsets of a completely q-convex domain are not completely strictly q-convex. However, in our case it is easy to construct from a q-convex function ρ such that

$$U \cap V = \{x \in V \,|\, \rho(x) < 0\}$$

for some neighborhood V of ∂U, a q-convex defining function defined in a neighborhood of $\overline{U}$. Indeed, it is enough to compose (η, ρ) with a good approximation of the maximum function (see [HL88, Definition 4.12]). It will give a $(q + 1)$-convex function since the positive eigenvalues of the complex Hessians of ρ and η are in the same directions. Therefore, U is completely strictly $(p-1)$-convex and we have the following solution for the dd^c-equation in symmetric bidegrees.

Theorem 3.4. *Let U be as above. If φ is a $\mathcal{C}^2$ (r,r)-form in a neighborhood of $\overline{U}$ with $k-p \leq r \leq k$, then there exists a continuous (r,r)-form ψ on U such that $dd^c\psi = dd^c\varphi$ and*

$$\|\psi\|_{\infty,U} \leq C\|dd^c\varphi\|_{\infty,U}$$

for some $C > 0$ independent of φ.

Proof. The proof follows closely the proof of Theorem 2.7 in [DNS08].

Without loss of generality, we can assume that φ is real and therefore $dd^c\varphi$ is also real. First, we solve the equation $d\xi = dd^c\varphi$ with estimates. Let W be a small neighborhood of $\overline{U}$, with the same geometric property and such that φ is defined on W. The maps A_θ defined in Section 2 give a homotopy $A : [0,1]\times W \to W$, $A(\theta,x) = A_\theta(x)$, between $A_1 = \mathrm{Id}$ and the projection A_0 of W on L. Since L has dimension $k-p$, A_0^* vanish identically on $(r+1,r+1)$-forms if $r \geq k-p$. Therefore, by homotopy formula (see e.g [BT82, p38]), there exists a form ξ on W such that $d\xi = dd^c\varphi$ and $\|\xi\|_{\infty,U} \lesssim \|dd^c\varphi\|_{\infty,U}$. Moreover, possibly by exchanging ξ by $(\xi+\overline{\xi})/2$, we can assume that $\xi = \Xi+\overline{\Xi}$ where Ξ is a $(r,r+1)$-form. As $d\xi$ is a $(r+1,r+1)$-form, it follows that $\overline{\partial}\Xi = 0$ and $d\xi = \partial\Xi + \overline{\partial\Xi}$. Therefore, by Theorem 3.2, Ξ is $\overline{\partial}$-exact and by Theorem 3.1, there exists a continuous (r,r)-form Ψ such that $\overline{\partial}\Psi = \Xi$ and $\|\Psi\|_{\infty,U} \lesssim \|\Xi\|_{\infty,U}$.

Finally, if $\psi = -i\pi(\Psi - \overline{\Psi})$ we have

$$dd^c\psi = \partial\overline{\partial}(\Psi - \overline{\Psi}) = \partial\Xi + \overline{\partial\Xi} = dd^c\varphi,$$

and

$$\|\psi\|_{\infty,U} \lesssim \|\Xi\|_{\infty,U} \lesssim \|dd^c\varphi\|_{\infty,U}.$$

□

4 Attracting speed

For R in $\mathscr{C}_p(U)$, we denote by R_n its normalized push-forward by f^n, i.e. $R_n := d^{-(k-p)n}(f^n)_*(R)$. To obtain (1.1), the first observation is that the norm of $R_n - \tau$, seen as a linear form on the space of continuous test $(k-p,k-p)$-forms, is bounded independently of n and R. Therefore, it is sufficient to establish (1.1) for $\alpha = 2$ and then apply interpolation theory between Banach spaces, see e.g. [Tri95], in order to obtain the general case.

Let denote by X the set of all real continuous $(k-p,k-p)$-forms ϕ on U such that $dd^c\phi = 0$ and $|\langle R-\tau,\phi\rangle| \leq 1$ for all $R \in \mathscr{C}_p(U)$. Observe that, since $f(U) \Subset U$, if ϕ is in X then $f^*(\phi)$ is defined on U where it is still a real continuous form with $dd^c(f^*(\phi)) = 0$. The set X is a truncated convex cone and the first part of the proof of Theorem 1.1 is to show that $d^{-(k-p)} f^*$ acts by contraction on it. This result is available without any assumption on $\| \wedge^{k-p+1} Df\|$. It is based on Lemma 2.1 and Harnack's inequality for harmonic functions.

Lemma 4.1. *There exists a constant* $0 < \lambda_1 < 1$ *such that for any* R *in* $\mathscr{C}_p(U)$, ϕ *in* X *and* n *in* $\mathbb{N}$ *we have*

$$|\langle R_n - \tau, \phi\rangle| \leq \lambda_1^n.$$

Proof. If R is in $\mathscr{C}_p(U)$ and ϕ in X, $R_1 := d^{-(k-p)} f_*(R)$ is in $\mathscr{C}_p(U')$ and we define the function $h_{R,\phi}$ on V by $h_{R,\phi}(\theta) := \langle R_{1,\theta} - \tau, \phi\rangle$, where $\theta \mapsto R_{1,\theta}$ is the structural disc described in Section 2. The definition of X implies that $|h_{R,\phi}| \leq 1$ on V, for all $R \in \mathscr{C}_p(U)$ and $\phi \in X$. Moreover, since R_1 is in $\mathscr{C}_p(U')$, it follows from Lemma 2.1 that all these functions are harmonic on V.

Now, observe that if we take $R = \tau$ then $h_{\tau,\phi}(1) = 0$ for all $\phi \in X$, since $d^{p-k} f_*\tau = \tau$. Hence, as $|h_{\tau,\phi}| \leq 1$ on V, Harnack's inequality says that there exists $0 \leq a < 1$ such that $|h_{\tau,\phi}(0)| \leq a$ for all ϕ in X. On the other hand, $R_{1,0}$ is a current independent of R. So, for all $R \in \mathscr{C}_p(U)$ and $\phi \in X$ we have $h_{R,\phi}(0) = h_{\tau,\phi}(0)$ and therefore $|h_{R,\phi}(0)| \leq a$. Once again, we deduce from Harnack's inequality there exists $0 < \lambda_1 < 1$, independent of R and ϕ, such that $|h_{R,\phi}(1)| \leq \lambda_1$ or equivalently

$$\left|\left\langle R_1 - \tau, \frac{\phi}{\lambda_1}\right\rangle\right| = |\langle R - \tau, \phi_1\rangle| \leq 1,$$

where $\phi_1 = d^{-(k-p)} f^*(\phi/\lambda_1)$. Moreover, ϕ_1 is defined on U and $dd^c\phi_1 = 0$. It follows that ϕ_1 is in X. Using the same arguments with ϕ_1 instead of ϕ gives that $|\langle R_1 - \tau, \phi_1\rangle| \leq \lambda_1$ which can be rewrite $|\langle R_2 - \tau, \phi\rangle| \leq \lambda_1^2$. Inductively, we obtain that $|\langle R_n - \tau, \phi\rangle| \leq \lambda_1^n$. □

Remark 4.2. *The constant* λ_1 *is not directly related to* f. *Indeed, it only depends on* V *i.e. on the size of* U *and the distance between* ∂U *and* $\partial f(U)$. *If* h *is the unique biholomorphism between* V *and the unit disc in* $\mathbb{C}$ *such that* $h(0) = 0$ *and* $h(1) = \alpha \in]0, 1[$ *then Harnack's inequality gives explicitly that we can take* $a = 2\alpha/(1 + \alpha)$ *and* $\lambda_1 = 4\alpha/(1 + \alpha)^2$.

In order to prove Theorem 1.1, we use Theorem 3.4 together with the assumption on $\| \wedge^{k-p+1} Df\|$ and Lemma 4.1.

If $\| \wedge^{k-p+1} Df(z)\| < 1$ on $\overline{U}$ then by continuity, there exists a constant $0 < \lambda_2 < 1$ such that $\|\wedge^{k-p+1} Df(z)\| < \lambda_2$ on U. Hence, if φ is a $(k-p, k-p)$-form of class $\mathcal{C}^2$, we have for $\varphi_i := d^{-i(k-p)}(f^i)^*(\varphi)$ with $i \in \mathbb{N}$

$$\|dd^c \varphi_i\|_{\infty,U} \lesssim \frac{\lambda_2^{2i}}{d^{i(k-p)}} \|\varphi\|_{\mathcal{C}^2}.$$

Here, the symbol $\lesssim$ means inequality up to a constant which only depends on our conventions and on U. By Theorem 3.4 with $r = k - p$, there exists a continuous $(k-p, k-p)$-form ψ_i on U such that

$$dd^c \psi_i = dd^c \varphi_i$$

and

$$\|\psi_i\|_{\infty,U} \lesssim \|dd^c \varphi_i\|_{\infty,U} \lesssim \frac{\lambda_2^{2i}}{d^{i(k-p)}} \|\varphi\|_{\mathcal{C}^2}.$$

We can now complete the proof of our main result.

End of the proof of Theorem 1.1. Let R be in $\mathscr{C}_p(U)$ and φ be a $(k-p, k-p)$-form of class $\mathcal{C}^2$. Without loss of generality, we can assume that φ is real. Let $0 \leq i \leq n$ be two arbitrary integers. We set $l := n - i$. If R_n, φ_i and ψ_i are defined as above then we have

$$\langle R_n - \tau, \varphi\rangle = \langle R_l - \tau, \varphi_i\rangle = \langle R_l - \tau, \varphi_i - \psi_i\rangle + \langle R_l - \tau, \psi_i\rangle,$$

since τ is invariant. On the one hand,

$$\langle R_l - \tau, \psi_i\rangle \lesssim \|\psi_i\|_{\infty,U} \lesssim \frac{\lambda_2^{2i}}{d^{i(k-p)}} \|\varphi\|_{\mathcal{C}^2}, \tag{4.1}$$

since R_l and τ are supported on U. On the other hand, observe that there exists a constant $M \geq 1$ independent of φ such that $\|d^{-(k-p)} f^*(\varphi)\|_\infty \leq M\|\varphi\|_\infty$. Therefore,

$$\begin{aligned}\|\varphi_i - \psi_i\|_{\infty,U} \leq M^i \|\varphi\|_\infty + \|\psi_i\|_{\infty,U} &\leq M^i \|\varphi\|_\infty + C\frac{\lambda_2^{2i}}{d^{i(k-p)}} \|\varphi\|_{\mathcal{C}^2} \\ &\lesssim M^i \|\varphi\|_{\mathcal{C}^2},\end{aligned}$$

and in particular

$$|\langle S - \tau, \varphi_i - \psi_i\rangle| \lesssim M^i \|\varphi\|_{\mathcal{C}^2},$$

for any S in $\mathscr{C}_p(U)$.

Moreover, $\varphi_i - \psi_i$ is a real continuous $(k-p, k-p)$-form on U and $dd^c(\varphi_i - \psi_i) = 0$. Hence, $(\varphi_i - \psi_i)/(CM^i \|\varphi\|_{\mathcal{C}^2})$ belongs to X where $C > 0$

is a constant depending only on U and on our conventions. It follows by Lemma 4.1 that

$$|\langle R_l - \tau, \varphi_i - \psi_i \rangle| \leq CM^i \|\varphi\|_{\mathcal{C}^2} \lambda_1^l. \tag{4.2}$$

To summarize, equations (4.1) and (4.2) imply that there are constants $0 < \lambda_1, \lambda_2 < 1$, and $M \geq 1$ such that

$$|\langle R_n - \tau, \varphi \rangle| \lesssim \|\varphi\|_{\mathcal{C}^2} \left(M^i \lambda_1^l + \frac{\lambda_2^{2i}}{d^{i(k-p)}} \right).$$

If $q \in \mathbb{N}$ is large enough then $M\lambda_1^q < 1$. Therefore, if we choose $n \simeq (q+1)i$, we obtain $l \simeq iq$ and

$$|\langle R_n - \tau, \varphi \rangle| \lesssim \|\varphi\|_{\mathcal{C}^2} \lambda^n,$$

where $\lambda := \max(\lambda_2^2 d^{-(k-p)}, M\lambda_1^q)^{1/(q+1)} < 1$. This estimate holds for arbitrary n in $\mathbb{N}$ and is uniform on φ and R. □

Remark 4.3. *In Theorem 1.1, it is enough to assume that* $\|\wedge^{k-p+1} Df(z)\| < d^{(k-p)/2}$ *on* $\overline{U}$. *Moreover, it is easy using small perturbations of a suitable polynomial map to construct examples with* $\| \wedge^{k-p+1} Df(z)\|$ *as small as we want on* $\overline{U}$.

5 Hyperbolicity of the equilibrium measure

In this section, we prove Theorem 1.2. Recall that the equilibrium measure associated to A is given by $\nu := \tau \wedge T^{k-p}$. It has maximal entropy on A equal to $(k-p)\log d$, [Din07]. On the other hand, we have the following powerful result, see [dT08] and [Dup09].

Theorem 5.1. *If the Lyapunov exponents of* ν *are ordered so that*

$$\chi_1 \geq \cdots \geq \chi_{a-1} > \chi_a \geq \cdots \geq \chi_k,$$

then

$$h(\nu) \leq (a-1)\log d + 2\sum_{i=a}^{k} \chi_i^+, \tag{5.1}$$

where $h(\nu)$ *denotes the entropy of* ν *and* $\chi_i^+ := \max(\chi_i, 0)$.

Now, let $1 \leq c \leq k$ be such that

$$\chi_1 \geq \cdots \geq \chi_c > 0 \geq \chi_{c+1} \geq \cdots \geq \chi_k.$$

If we take $a = c+1$ in Theorem 5.1, we obtain $h(\nu) \leq c\log d$. Since $h(\nu) = (k-p)\log d$, it follows that $c \geq (k-p)$. It means there are at least $k-p$ strictly positive Lyapunov exponents. Moreover, if we have equality, $c = k - p$, Theorem 5.1 applied to the smallest a such that $\chi_a = \chi_c$ gives

$$(k-p)\log d = h(\nu) \leq (a-1)\log d + 2(k-p-a+1)\chi_c.$$

Hence, $\chi_c \geq (\log d)/2$. Note that in this part we do not need the assumption on $\| \wedge^{k-p+1} Df\|$.

It remains to prove that the assumptions of Theorem 1.1 imply that $c \leq k-p$ and $\chi_{c+1} < -(k-p)(\log d)/2$. It is not hard to deduce form Oseledec theorem [Ose68] that the sum of the q largest Lyapunov exponents verifies

$$\chi_1 + \cdots + \chi_q = \lim_{n\to\infty} \frac{1}{n} \log \| \wedge^q Df^n(z)\|,$$

for ν-almost all z. Moreover, we have

$$\| \wedge^q Df^{n+m}(z)\| \leq \| \wedge^q Df^n(z)\| \| \wedge^q Df^m(f^n(z))\|.$$

Therefore, it follows that

$$\| \wedge^q Df^n(z)\| \leq (\max_{z\in U} \| \wedge^q Df(z)\|)^n$$

and

$$\chi_1 + \cdots + \chi_q \leq \log \max_{z\in U} \| \wedge^q Df(z)\| =: \gamma.$$

Hence, if $\| \wedge^{k-p+1} Df(z)\| < 1$ on $\overline{U}$ then

$$\chi_1 + \cdots + \chi_{k-p+1} \leq \gamma < 0.$$

Therefore, $c \leq k-p$ and we have seen above that in this case $c = k-p$ and $\chi_c \geq (\log d)/2$. Finally, we have

$$\gamma \geq \chi_1 + \cdots + \chi_{k-p} + \chi_{k-p+1} \geq \frac{k-p}{2}\log d + \chi_{k-p+1},$$

which implies

$$\chi_{k-p+1} \leq \gamma - \frac{k-p}{2}\log d.$$

Remark 5.2. *Theorem 5.1 with $a = 1$ implies the Ruelle inequality, i.e.*

$$\chi_1 + \cdots + \chi_c \geq \frac{k-p}{2}\log d.$$

Therefore, it is enough to assume that $\| \wedge^{k-p+1} Df(z)\| < d^{(\frac{k-p}{2})(\frac{k-p+1}{k})}$ on $\overline{U}$ since

$$\chi_1 + \cdots + \chi_{k-p+1} \geq \frac{k-p+1}{c}(\chi_1 + \cdots + \chi_c),$$

if $c \geq k-p+1$.

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