SPARSITY OF POSTCRITICALLY FINITE MAPS OF $\mathbb{P}^k$ AND BEYOND: A COMPLEX ANALYTIC APPROACH

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Abstract. An endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ of degree $d \geq 2$ is said to be postcritically finite (or PCF) if its critical set $\mathrm{Crit}(f)$ is preperiodic, i.e. if there are integers $m > n \geq 0$ such that $f^m(\mathrm{Crit}(f)) \subseteq f^n(\mathrm{Crit}(f))$. When $k \geq 2$, it was conjectured in [IRS] that, in the space End_d^k of all endomorphisms of degree d of $\mathbb{P}^k$, such endomorphisms are not Zariski dense. We prove this conjecture. Further, in the space Poly_d^2 of all regular polynomial endomorphisms of degree $d \geq 2$ of the affine plane $\mathbb{A}^2$, we construct a dense and Zariski open subset where we have a uniform bound on the number of preperiodic points lying in the critical set.

The proofs are a combination of the theory of heights in arithmetic dynamics and methods from real dynamics to produce open subsets with maximal bifurcation.

Contents

Introduction

Let $\pi : \mathcal{X} \to S$ be a family of complex projective varieties, where S is a smooth complex projective variety, and let $\mathcal{L}$ be a nef and relatively ample line bundle on $\mathcal{X}$. We let $f : \mathcal{X} \dashrightarrow \mathcal{X}$ be a rational map such that $(\mathcal{X}, f, \mathcal{L})$ is a family of polarized endomorphisms of degree $d \geq 2$ over a Zariski open subset S^0 of S, i.e. for all $t \in S^0(\mathbb{C})$, $X_t := \pi^{-1}\{t\}$ is smooth, $L_t := \mathcal{L}|_{X_t}$ is ample and $f_t^* L_t \simeq L_t^{\otimes d}$. If $\mathcal{X}^0 = \pi^{-1}(S^0)$, the family $\mathcal{X}^0 \to S^0$ is the *regular part* of $(\mathcal{X}, f, \mathcal{L})$. The purpose of the article is to study maximal instability phenomena in both complex and arithmetic dynamics, each viewpoint giving deep insights into the other.

From the arithmetic viewpoint, we are mainly interested in the notion of canonical height of a subvariety. Such height is a function meant to measure the arithmetic dynamical complexity of the orbit of the subvariety. Studying such objects in family, we are

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particularly interested in two cases: the universal family

$$f : \mathbb{P}^k_{\mathscr{M}_d^k} \longrightarrow \mathbb{P}^k_{\mathscr{M}_d^k}$$

where $\mathscr{M}_d^k$ is the moduli space of degree d and also the universal family

$$f : \mathbb{P}^k_{\mathscr{P}_d^k} \longrightarrow \mathbb{P}^k_{\mathscr{P}_d^k}$$

where $\mathscr{P}_d^k$ is the moduli space of regular polynomial endomorphisms of the affine space $\mathbb{A}^k$, see § 2.1 for more details. In both cases, the family of subvarieties we consider is the critical set. More precisely, we

- show that this height is in fact a moduli height on a Zariski open set U of $\mathscr{M}_d^k$.
- use that height to show that postcritically finite maps – PCF maps for short – (see below) are not Zariski dense in $\mathscr{M}_d^k$ nor in $\mathscr{P}_d^2$.
- prove a uniform bound on the number of preperiodic critical points for regular polynomial endomorphisms whose conjugacy lies in a Zariski open set of $\mathscr{P}_d^2$.

The complex analytic viewpoint is essential in that process to

- show that the support of the bifurcation measure (see below) has non-empty interior in both $\mathscr{M}_d^k$ and $\mathscr{P}_d^2$.
- prove that the correspondence between an endomorphism in $\mathscr{M}_d^k$ (or $\mathscr{P}_d^2$) and the collection of the multipliers of its periodic points is finite-to-one outside a Zariski closed set.

We are strongly inspired by the recent results on families of abelian varieties where similar type of results have been established, as well as by the recent uniform bounds on the number of common preperiodic points for rational maps of $\mathbb{P}^1$, initiated by DeMarco, Krieger and Ye [DKY1, DKY2] in the cases of flexible Lattès maps and quadratic polynomials, and developed since then by Mavraki and Schmidt [MS] and DeMarco and Mavraki [DM]. Concerning families of abelian varieties, they naturally fall in the setting of family of polarized endomorphisms when taking the multiplication by $[n]$ morphism. In particular, we used ideas coming from the work of Gao-Habegger [GH] where the Geometric Bogomolov conjecture is proved on curves (see also [CGHX, XY] for proofs in the general case), the work of Dimitrov-Gao-Habegger [DGH] where a uniform bound on the number of rational points of a curve C, defined over a number field, inside its Jacobian is established (Uniform Mordell-Lang) and the works of Kühne [K], generalized by Yuan in arbitrary characteristic [Yu], and Gao-Ge-Kühne [GGK] where the Uniform Mordell-Lang Conjecture is generalized to arbitrary subvariety of an abelian variety.

A crucial point in our work is to link the notion of dynamical stability in complex dynamics, which can be characterized by positive closed currents, with the notion of dynamical height. In [GV], relying on the theory of DSH functions of Dinh and Sibony [DS4], the first and third authors established such link for the $(1,1)$ *bifurcation current* of a family of subvarieties, here we need to deal with the *bifurcation measure*, which measures higher bifurcation phenomena. Let us explain those terms.

Let ω be a smooth positive form representing the first Chern class $c_1(\mathcal{L})$ on $\mathcal{X}$. As $f^*\mathcal{L} \simeq \mathcal{L}^{\otimes d}$ on $\mathcal{X}^0$, there is a smooth function $g : \mathcal{X}^0 \to \mathbb{R}$ such that $d^{-1}f^*\omega = \omega + dd^c g$ as forms on $\mathcal{X}^0$. In particular, the following limit exists as a closed positive $(1,1)$-current with continuous potential on the quasi-projective variety $\mathcal{X}^0(\mathbb{C})$:

$$\widehat{T}_f := \lim_{n\to\infty} \frac{1}{d^n}(f^n)^*(\omega),$$

and can be written as $\widehat{T}_f = \omega + dd^c g_f$, where g_f is continuous on $\mathcal{X}^0(\mathbb{C})$. The current $\widehat{T}_f$ is the *fibered Green current* of f (note that for abelian varieties, $\widehat{T}_f$ is the Betti form). Let $\mathcal{Y} \to S$ be a family of subvarieties of $\mathcal{X}$, i.e. $\mathcal{Y}$ is a subvariety of $\mathcal{X}$ and $\pi|_{\mathcal{Y}} : \mathcal{Y} \to S$ of π is flat over S^0. If q is the relative dimension of $\mathcal{Y}$, for $1 \leq m \leq \dim S$, the *m-bifurcation current* of $(\mathcal{X}, f, \mathcal{L}, \mathcal{Y})$ can be defined on $S^0(\mathbb{C})$ as

$$T^{(m)}_{f,\mathcal{Y}} := (\pi_{[m]})_* \left(\widehat{T}^{m(\dim Y_\eta + 1)}_{f^{[m]}} \wedge [\mathcal{Y}^{[m]}] \right),$$

where Y_η is the generic fiber of $\mathcal{Y}$, $\pi_{[m]} : \mathcal{X}^{[m]} \to S$ is the m-fiber product of $\mathcal{X}$, and $f^{[m]}$ is the map induced by the fiberwise diagonal action of f. The *bifurcation measure* of $(\mathcal{X}, f, \mathcal{L}, \mathcal{Y})$ is then

$$\mu_{f,\mathcal{Y}} := T^{(\dim S)}_{f,\mathcal{Y}}.$$

We now focus on the case of a family of rational maps of $\mathbb{P}^k(\mathbb{C})$, parametrized by a projective variety S. In this case, the regular part is $\mathcal{X}^0 = \mathbb{P}^k \times S^0$, where S^0 is a Zariski open subset of S. We then are interested in the bifurcation of the critical set $\mathrm{Crit}(f) := \{(z,t) \in \mathbb{P}^k \times S^0 \,:\, \det(D_z f_t) = 0\}$. So, the bifurcation measure is

$$\mu_{f,\mathrm{Crit}} := T^{(\dim S)}_{f,\mathrm{Crit}(f)} = (\pi_{[\dim S]})_* \left(\widehat{T}^{k(\dim S)}_{f^{[\dim S]}} \wedge [\mathrm{Crit}(f)^{[\dim S]}] \right),$$

since $\mathrm{Crit}(f)$ is a hypersurface of $\mathbb{P}^k \times S^0$.

When $k = 1$, the bifurcation current has been introduced by DeMarco [De] and the bifurcation measure by Bassanelli-Berteloot [BB1]. For families of endomorphisms of $\mathbb{P}^k$, the bifurcation current has been introduced by Bassanelli-Berteloot [BB1]. In this higher dimensional setting, Berteloot-Bianchi-Dupont showed it is the appropriate tool for studying bifurcations in the important work [BBD] and the bifurcation measure was first considered by Astorg and Bianchi [AB] in the very particular case of families of polynomial skew-product.

It is an important question in complex dynamics to understand what kind of phenomena these currents (or this measure) actually characterize. One way to explore this question is to prove that the measure $\mu_{f,\mathrm{Crit}}$ equidistributes specific type of dynamical behaviors ([BB2, DF, FG, GOV1]).

We now come to stating precise results. Define the critical height of a degree d endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ defined over a number field as the canonical height of f evaluated at the critical set of f:

$$h_{\mathrm{crit}}(f) := \widehat{h}_f(\mathrm{Crit}(f))$$

and remark that this quantity depends only on the conjugacy class. In particular, this defines a function

$$h_{\mathrm{crit}} : \mathscr{M}^k_d(\bar{\mathbb{Q}}) \to \mathbb{R}_+.$$

Our first result here is the following

Theorem A (The critical height is a moduli height)**.** *The critical height h_{crit} of the moduli space $\mathscr{M}^k_d$ of degree d of endomorphisms of $\mathbb{P}^k$ is an ample height on a non-empty Zariski open subset U of $\mathscr{M}^k_d$, i.e. for any ample line bundle M on a projective model of $\mathscr{M}^k_d$, there are constants $C_1, C_2 > 0$ and $C_3, C_4 \in \mathbb{R}$ such that*

$$C_1 \cdot h_M([f]) + C_3 \leq h_{\mathrm{crit}}([f]) \leq C_2 h_M([f]) + C_4,$$

for all $[f] \in \mathscr{M}^k_d(\bar{\mathbb{Q}})$. Moreover, a subvariety Z is an irreducible component of $\mathscr{M}^k_d \setminus U$ if and only if the bifurcation measure $\mu_{f,\mathrm{Crit},Z}$ of the family induced by Z vanishes.

For $k = 1$, this result is due to Ingram [I1] (see also [GOV2]). For $k \geq 1$, Ingram also proved explicit versions of the above theorem for specific families using convenient parametrizations (e.g. [I2, I3]). In dimension 1, McMullen's result [Mc] implies that the algebraic subvariety where we do not have the inequality in Theorem A is exactly the flexible Lattès family. Characterizing that subvariety in higher dimension is one of the main questions in bifurcation theory in higher dimension.

In order to prove Theorem A, we follow Gao and Habbeger and Dimitrov in the abelian case [GH, DGH] to prove an estimate in a family with positive (suitable) height which compares the height of a parameter with the heights of generic point in the m-fiber product of m-fiber product of $\mathcal{Y}$. Our arguments are based on the early work [GV] of the first and third authors (see Theorem 5.4). We then use notably Zhang inequalities [Zha1] to conclude.

Let $k \geq 1$. Let End_d^k denote the set of endomorphisms $f : \mathbb{P}^k(\mathbb{C}) \to \mathbb{P}^k(\mathbb{C})$ of degree d (in homogeneous coordinates, f is the data of $k+1$ homogeneous polynomials with no common factor and the same degree d). Such f is *postcritically finite* (PCF for short) if its postcritical set

$$\mathrm{PC}(f) := \bigcup_{n \geq 1} f^n(\mathrm{Crit}(f))$$

is an algebraic subvariety of $\mathbb{P}^k$, where $\mathrm{Crit}(f) = \{z \in \mathbb{P}^k(\mathbb{C}) : \det(D_z f) = 0\}$ is its *critical set*. In dimension 1, the critical set is a finite set of cardinality $2d-2$ so, for all n, $f^n(\mathrm{Crit}(f))$ is again a finite set of cardinality $2d-2$ (counting the multiplicity), so PCF maps are not so hard to exhibit and it turns out that PCF maps are in fact Zariski dense (e.g. [DF, BE2, GOV1, Ga3]). In higher dimension, the algebraic hypersurface $\mathrm{Crit}(f)$ has positive dimension, hence it is not finite, and this fact is responsible for several new phenomena arising in complex dynamics in several variables.

Nevertheless, such maps, which descend to $\mathscr{M}_d^k$, are of the utmost arithmetic importance since they satisfy $h_{\mathrm{crit}}(f) = 0$ (in dimension 1, it is known that the converse is true by Northcott's property, for $k > 1$, this is an open and important problem). This raises the following theorem:

Theorem B (Sparsity of PCF maps)**.** *Fix two integers $k, d \geq 2$. There exists a strict subvariety $V_d^k \subsetneq \mathrm{End}_d^k$ such that any PCF endomorphism f is contained in V_d^k.*

Such result was conjectured by Ingram, Ramadas and Silverman in [IRS] where they show that $\{f \in \mathrm{End}_d^k,\ f^n(\mathrm{Crit}(f)) = f^m(\mathrm{Crit}(f))\}$ is not Zariski dense for $m \in \{0, 1, 2\}$ and $d \geq 3$. Our approach is inspired by Kühne's Relative Equidistribution [K] on families of abelian varieties defined over a number field $\mathbb{K}$ which we generalize to the setting of families of polarized endomorphisms using the arithmetic equidistribution theorem of the first author [Ga3] and Theorem A (see also [YZ]). Thus, if PCF maps were Zariski dense, they would equidistribute the bifurcation measure. In order to get a contradiction, we inject the following crucial theorem working at the complex place.

Theorem C (Robust strong bifurcations)**.** *Fix two integers $k, d \geq 2$. There exists an non-empty analytic open subset $\Omega \subset \mathscr{M}_d^k(\mathbb{C})$ (resp. $\Omega \subset \mathscr{P}_d^2$) such that*

- *the open set Ω is contained in* $\mathrm{supp}(\mu_{f,\mathrm{Crit}})$,
- *the open set Ω contains no PCF conjugacy class.*

Observe that we do not prove the theorem for the moduli space $\mathscr{P}_d^k$ when $k \geq 3$. This is a technical issue that simply comes from the fact that our proof of the generic finiteness of the multiplier maps only works on $\mathscr{M}_d^k$ and $\mathscr{P}_d^2$. The same result probably holds on $\mathscr{P}_d^k$

in all dimensions but our main motivation for the polynomial case is Theorem D whose counterpart (see Theorem 7.2) is weaker when $k \geq 3$.

In dimension 1, Montel theorem easily implies that the bifurcation locus (i.e. the support of the bifurcation current) has empty interior. In higher dimension, Bianchi and the second author first gave an example (the Desboves family) where this is not the case [BT]. In general, the fact that the support of the bifurcation current has non-empty interior is due to the seminal work of Dujardin [Du], whose ideas were improved by the second author in [T] to produce open sets of bifurcation in other situations. A different approach, due to Biebler, was to construct an open set of bifurcations around Lattès maps [Bie]. Turning to the bifurcation measure, the firsts to prove the non-emptiness of the support of the bifurcation measure were Astorg and Bianchi [AB] in the very particular case of the family of polynomial skew-products (with given base dynamics satisfying certain additional assumptions) of $\mathbb{C}^2$.

An important ingredient in the proof of Theorem C is a mechanism called *blender* in smooth dynamics. It was introduced by Bonatti-Díaz in [BD3] to obtain new examples of robustly transitive diffeomorphisms. Since then, it was used in a wide range of contexts in real dynamics (see e.g. [RHRHTU], [BD5] or [Be]). A remarkable feature about blenders is that they are much easier to construct than other mechanisms given robust intersections (like the Newhouse phenomenon). A characteristic of particular importance in the rigid setting of holomorphic dynamics where they were first introduced by Dujardin in [Du].

Roughly speaking, in our context a blender for a map f is a repelling hyperbolic set (typically a Cantor set) that intersects an open family of (local) hypersurfaces and this property persists for small perturbations of f. Dujardin constructed in [Du] a map with a blender for which a part of the postcritical set belongs to the associated family of hypersurfaces. This provides a robust intersection between the blender and the postcritical set which turns out to be sufficient to have an open set in the bifurcation locus. The same strategy can be followed in order to prove Theorem C except that instead of a single intersection we need as many as possible (i.e. the dimension of the moduli space) independent intersections, i.e. which satisfy the transversality condition of Definition 1.10. To that end, we consider a map f with a blender $\Lambda(f)$ and a saddle point $p(f)$ whose unstable manifold intersects robustly $\Lambda(f)$. Observe that in the terminology of smooth dynamics, this corresponds in our non-invertible context to a *robust heterodimensional cycles* (see [BD4] for the interplays between these cycles and blenders in the C^1-setting). As the critical set has to intersect the stable manifold of $p(f)$, the inclination lemma gives infinitely many intersections between $\Lambda(f)$ and the postcritical set of f. All the difficulty in the proof is to check that they provide enough independent intersections. This brings us to prove (very) partial generalizations to higher dimension of several results from one-dimensional complex dynamics, like extension of local conjugacies [BE1], the rigidity of stable algebraic families [Mc] or the fact that multipliers of periodic points give (generically) local coordinates in the moduli space [Mc, Go, JX].

In the particular case of regular polynomial endomorphisms of the affine plane, we consider bifurcations of the finite part of the critical set, i.e. of the closure C_f in $\mathbb{P}^2$ of the set $\{z \in \mathbb{C}^2, \det(D_z f) = 0\}$. In this case, the non-negativity of the Green function at every place allows us to prove the following uniform result.

Theorem D (Uniformity)**.** *Fix an integer $d \geq 2$. There exists a constant $B(d) \geq 1$ and a non-empty Zariski open subset $U \subset \mathrm{Poly}_d^2$ such for any $f \in U(\mathbb{C})$, we have*

$$\#\mathrm{Preper}(f) \cap C_f \leq B(d).$$

As mentioned before, similarly flavored uniform results already exist in complex dynamics and are a very important source of inspiration for us: the quotient $\tilde{S} := (\mathrm{End}_d^1 \times \mathrm{End}_d^1)/\mathrm{PGL}(2)$ by the diagonal action by conjugacy is a quasiprojective variety. On a suitable subvariety S of $\tilde{S}$, one wants to show that outside a Zariski closed set of S, then all pairs of rational maps $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ will have at most B preperiodic points in common. This is a statement very similar to Theorem D where one considers $\mathcal{Y}$ to be the fibered diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ instead of the critical set. Then, such a result was first shown by DeMarco, Krieger and Ye in the Legendre family [DKY1] and in the quadratic family [DKY2]. Then, Mavraki and Schmidt proved it in the case of any algebraic curve S [MS]. Finally, DeMarco and Mavraki have shown very recently the optimal result that there is a uniform bound B, depending only on the degree d, so that for a Zariski open and dense set in the space of all pairs of rational maps $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ with degree d, f and g have at most B preperiodic points in common [DM].

To prove Theorem D, we show in Theorem 7.7 that there is a height gap (there exists a $\varepsilon > 0$ such that all points of canonical height $\leq \varepsilon$ are contained in a Zariski closed proper subset of the fibered critical locus). For that, we follow the idea of Gao, Ge and Kühne on abelian varieties [GGK] (first introduced by Ullmo [U] and Zhang [Zha2]) to overfiber the dynamics (see also [MS] and [DM] where this strategy is also used). The fact that local heights are all positive in the polynomial setting allows us to get the bound from the complex place.

Remark. Thanks to [AB, Corollary 1.4], the same statement as that of Theorem D holds in the family of all degree d polynomial skew-products with given postcritically finite base $p \in \bar{\mathbb{Q}}[z]$, as soon as p is neither conjugated to z^d nor to the degree d Chebyshev polynomial, see § 7.4.

Organization of the article. Section 1 is devoted to the construction of bifurcation currents and the corresponding volume we will need to construct the m-order canonical height. The proof of Theorem C occupies the next three sections. In Section 2, we establish that the eigenvalues of the periodic points determine a conjugacy class up to finitely many choices generically in $\mathscr{M}_d^k$ and $\mathscr{P}_d^2$. In Section 3, we prove that if an open subset of $\mathscr{M}_d^k$ or $\mathscr{P}_d^k$ satisfies a certain set of assumptions and is not contained in the support of the bifurcation measure then it has to contain lots of families where the eigenvalues of most of the periodic points are constant. Open subsets verifying this large set of assumptions are constructed in Section 4, for all $k \geq 2$ and $d \geq 2$. In Section 5, we prove important height inequalities, and in particular Theorem A. Section 6 is devoted to the proof of the needed Relative Equidistribution theorem. In Section 7, we prove Theorems B and D.

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1. The dynamical volumes of a family of subvarieties

1.1. **The canonical height over a function field of characteristic zero.** In the whole section, we let $(\mathcal{X}, f, \mathcal{L})$ be a family of polarized endomorphisms of degree $d \geq 2$ with regular part $\mathcal{X}^0 \to S^0$, and let $\mathcal{Y} \subsetneq \mathcal{X}$ be a subvariety such that $\pi(\mathcal{Y}) = S$ and let $S^0_{\mathcal{Y}}$ be the maximal Zariski open subset of S^0 such that $\pi|_{\mathcal{Y}}$ is flat and projective over $S^0_{\mathcal{Y}}$.

Recall that the fibered Green current $\widehat{T}_f$ is the limit in the sense of currents on $\mathcal{X}^0(\mathbb{C})$ of the sequence $(d^{-n}(f^n)^*\omega)_n$, where ω is any smooth form representing the cohomology class $c_1(\mathcal{L}) \in H^{1,1}(\mathcal{X}(\mathbb{C}), \mathbb{R})$. The current $\widehat{T}_f$ is a closed positive $(1,1)$-current of finite mass on $\mathcal{X}^0(\mathbb{C})$ and, for any $\lambda \in S^0(\mathbb{C})$, the slice $T_\lambda := \widehat{T}_f|_{X_\lambda}$ is the Green current of f_λ, see e.g. [GV] for more details. Furthermore, if $k := \dim X_\lambda$, we let $\mu_\lambda := \deg_{L_\lambda}(X_\lambda)^{-1} \cdot T_\lambda^k$ be the unique maximal entropy measure of f_λ, we call its support the *small Julia set*, and denote it by J_k (or $J_k(f_\lambda)$ to stress the dependence of f_λ) [BD6, DS4].

Definition 1.1. *We say the tuple $(\mathcal{X}, f, \mathcal{L}, \mathcal{Y})$ is a* dynamical pair *parametrized by S and with regular part $S^0_{\mathcal{Y}}$.*

Let $\mathcal{M}$ be an ample $\mathbb{Q}$-line bundle on S. Let Y_η be the generic fiber of the family $\mathcal{Y} \to S^0_{\mathcal{Y}}$. Following [Zha1], we define

$$\widehat{h}_{f_\eta}(Y_\eta) := \lim_{n\to\infty} d^{-n(\dim Y_\eta+1)} \frac{\left((f^n)_*\{\mathcal{Y}\} \cdot c_1(\mathcal{L})^{\dim Y_\eta+1} \cdot c_1(\pi^*\mathcal{M})^{\dim S-1}\right)}{(\dim Y_\eta + 1)\deg_{Y_\eta}(L_\eta)}.$$

The next lemma follows from [GV]:

Lemma 1.2. *For any $\mathcal{Y}$ as above, $\widehat{h}_{f_\eta}(Y_\eta)$ is well-defined and satisfies $\widehat{h}_{f_\eta}((f_\eta)_*(Y_\eta)) = d\widehat{h}_{f_\eta}(Y_\eta)$. In addition, we can compute $\widehat{h}_{f_\eta}(Y_\eta)$ as*

$$\widehat{h}_{f_\eta}(Y_\eta) = \frac{1}{(\dim Y_\eta + 1)\deg_{Y_\eta}(L_\eta)} \int_{\mathcal{X}^0(\mathbb{C})} \widehat{T}_f^{\dim Y_\eta+1} \wedge [\mathcal{Y}] \wedge (\pi^*\omega_S)^{\dim S-1},$$

where ω_S is any smooth form representing $c_1(\mathcal{M})$.

Proof. Let $q := \dim Y_\eta$ and $p := \dim S$. The fact that it is well-defined and the formula relating the limit of $d^{-n(q+1)}\left((f^n)_*\{\mathcal{Y}\} \cdot c_1(\mathcal{L})^{q+1} \wedge c_1(\pi^*\mathcal{M})^{p-1}\right)$ with $\widehat{T}_f^{q+1} \wedge [\mathcal{Y}] \wedge (\pi^*\omega_S)^{p-1}$ are contained in [GV, Theorem B]. We then can compute

$$\begin{aligned}
\widehat{h}_{f_\eta}((f_\eta)_*(Y_\eta)) &= \frac{1}{(q+1)\deg_{Y_\eta}(f_\eta^* L_\eta)} \int_{\mathcal{X}^0(\mathbb{C})} \widehat{T}_f^{q+1} \wedge (f_*[\mathcal{Y}]) \wedge (\pi^*\omega_S)^{p-1} \\
&= \frac{1}{(q+1)d^q \deg_{Y_\eta}(L_\eta)} \int_{\mathcal{X}^0(\mathbb{C})} \widehat{T}_f^{q+1} \wedge (f_*[\mathcal{Y}]) \wedge (\pi^*\omega_S)^{p-1} \\
&= \frac{1}{(q+1)d^q \deg_{Y_\eta}(L_\eta)} \int_{\mathcal{X}^0(\mathbb{C})} \left(f^*\left(\widehat{T}_f^{q+1} \wedge (\pi^*\omega_S)^{p-1}\right)\right) \wedge [\mathcal{Y}] \\
&= \frac{d^{q+1}}{(q+1)d^q \deg_{Y_\eta}(L_\eta)} \int_{\mathcal{X}^0(\mathbb{C})} \widehat{T}_f^{q+1} \wedge [\mathcal{Y}] \wedge (\pi^*\omega_S)^{p-1} = d\widehat{h}_{f_\eta}(Y_\eta),
\end{aligned}$$

where we used that $\dim Y_\eta = q$ and $\pi \circ f = \pi$. $\square$

In particular, the last part of the lemma states that the height $\widehat{h}_{f_\eta}(Y_\eta)$ is > 0 if and only if the bifurcation current $T_{f,\mathcal{Y}} := \pi_* \left(\widehat{T}_f^{\dim Y_\eta+1} \wedge [\mathcal{Y}]\right)$ is not identically zero since

$$\int_{\mathcal{X}^0(\mathbb{C})} \widehat{T}_f^{\dim Y_\eta+1} \wedge [\mathcal{Y}] \wedge (\pi^*\omega_S)^{\dim S-1} = \int_{S^0(\mathbb{C})} \pi_*(\widehat{T}_f^{\dim Y_\eta+1} \wedge [\mathcal{Y}]) \wedge \omega_S^{\dim S-1}.$$

1.2. **The higher bifurcation currents of a pair.** As above, let $(\mathcal{X}, f, \mathcal{L}, \mathcal{Y})$ be a dynamical pair parametrized by S with regular part $S^0_{\mathcal{Y}}$.

Let $\mathcal{M}$ be an ample $\mathbb{Q}$-line bundle on S of volume 1. For any $m \geq 1$, let $\mathcal{X}^{[m]} := \mathcal{X} \times_S \cdots \times_S \mathcal{X}$ and $\mathcal{Y}^{[m]} := \mathcal{Y} \times_S \cdots \times_S \mathcal{Y}$ be the respective m-fiber power of $\mathcal{X}$ and $\mathcal{Y}$. Denote also by $\pi_m : \mathcal{X}^{[m]} \to S$ the morphism induced by π. We define $f^{[m]}$ as

$$f^{[m]}(x) = (f_t(x_1), \dots, f_t(x_m)), \quad x = (x_1, \dots, x_m) \in X_t^m = \pi_{[m]}^{-1}\{t\}.$$

For any $1 \leq j \leq m$, we let $p_j : \mathcal{X}^{[m]} \to \mathcal{X}$ be the projection onto the j-th factor of the fiber product and $\mathcal{L}^{[m]} := p_1^*\mathcal{L} + \cdots + p_m^*\mathcal{L}$. By construction and using $\widehat{T}_f^{\dim X_\eta+1} = 0$, we have

$$\widehat{T}_{f^{[m]}} = p_1^*(\widehat{T}_f) + \cdots + p_m^*(\widehat{T}_f) \quad \text{and} \quad \widehat{T}_{f^{[m]}}^{m \dim X_\eta} = C(m, \dim X_\eta) \bigwedge_{j=1}^m p_j^* \left(\widehat{T}_f^{\dim X_\eta}\right), \tag{1}$$

where $C(m, \dim X_\eta) := \prod_{j=1}^m \binom{j \dim X_\eta}{\dim X_\eta}$. We define higher bifurcation currents as follows:

Definition 1.3. *For $1 \leq m \leq \dim S$, the m-*bifurcation current *of $(\mathcal{X}, f, \mathcal{L}, \mathcal{Y})$ is the closed positive (m,m)-current on $S^0_{\mathcal{Y}}(\mathbb{C})$ given by*

$$T_{f,\mathcal{Y}}^{(m)} := (\pi_{[m]})_* \left(\widehat{T}_{f^{[m]}}^{m(\dim Y_\eta+1)} \wedge [\mathcal{Y}^{[m]}]\right).$$

The bifurcation measure *of $(\mathcal{X}, f, \mathcal{L}, \mathcal{Y})$ is*

$$\mu_{f,\mathcal{Y}} := T_{f,\mathcal{Y}}^{(\dim S)}.$$

We give basic properties of those currents.

Proposition 1.4. *The following properties hold*

(1) For any $1 \leq m \leq \dim S$ and any $j \leq m$

$$T_{f,\mathcal{Y}}^{(m)} \geq T_{f,\mathcal{Y}}^{(j)} \wedge T_{f,\mathcal{Y}}^{(m-j)},$$

(2) For all m, $T_{f,\mathcal{Y}}^{(m)} \neq 0$ implies $T_{f,\mathcal{Y}}^{(m-1)} \neq 0$. Similarly, for all $m \geq \dim S$,

$$(\pi_{[m]})_* \left(\widehat{T}_{f^{[m]}}^{\dim \mathcal{Y}^{[m]}} \wedge [\mathcal{Y}^{[m]}]\right) \geq \mu_{f,\mathcal{Y}}.$$

(3) if $\dim Y_\eta = \dim X_\eta - 1$, we have

$$\mu_{f,\mathcal{Y}} = \prod_{j=1}^{\dim S} \binom{j \dim X_\eta}{\dim X_\eta} \cdot \left(T_{f,\mathcal{Y}}^{(1)}\right)^{\wedge \dim S},$$

and for any $m \geq \dim S$, there is a constant $C_m \geq 1$ such that

$$(\pi_{[m]})_* \left(\widehat{T}_{f^{[m]}}^{\dim \mathcal{Y}^{[m]}} \wedge [\mathcal{Y}^{[m]}]\right) = C_m \cdot \mu_{f,\mathcal{Y}}.$$

Proof. Fix $1 \leq m \leq \dim S$. For the sake of simplicity, let us only consider the case where $j = 1$. Let $p_j : \mathcal{X}^{[m]} \to \mathcal{X}$ be the projection onto the j-th factor and $\pi_{[m]} : \mathcal{X}^{[m]} \to S$ be the canonical projection (so that $\pi_{[1]}$ is the projection $\mathcal{X} \to S$). Then

$$\widehat{T}_{f^{[m]}}^{m(\dim Y_\eta+1)} = \left(p_1^*(\widehat{T}_f) + \cdots + p_m^*(\widehat{T}_f)\right)^{m(\dim Y_\eta+1)} \geq \bigwedge_{j=1}^m p_j^* \left(\widehat{T}_f^{\dim Y_\eta+1}\right)$$

as $(p_j^*\widehat{T}_f)^{\dim Y_\eta+1} = p_j^*(\widehat{T}_f^{\dim Y_\eta+1})$. Using the equality,

$$[\mathcal{Y}^{[m]}] = \bigwedge_{j=1}^m p_j^*([\mathcal{Y}]),$$

we deduce that for a positive test form ϕ of bidimension (m, m) on $S^0_{\mathcal{Y}}$,

$$\begin{aligned}
\langle T^{(m)}_{f,\mathcal{Y}}, \phi\rangle &= \langle (\pi_{[m]})_* \left(\widehat{T}_{f^{[m]}}^{m(\dim Y_\eta+1)} \wedge [\mathcal{Y}^{[m]}]\right), \phi\rangle \\
&= \langle \widehat{T}_{f^{[m]}}^{m(\dim Y_\eta+1)} \wedge [\mathcal{Y}^{[m]}], \pi^*_{[m]}\phi\rangle \\
&\geq \left\langle \bigwedge_{j=1}^m p_j^* \left(\widehat{T}_f^{\dim Y_\eta+1} \wedge [\mathcal{Y}]\right), \pi^*_{[m]}\phi \right\rangle \\
&\geq \left\langle \widehat{T}_f^{\dim Y_\eta+1} \wedge [\mathcal{Y}] \wedge (p_1)_* \left(\bigwedge_{j=2}^m p_j^* \left(\widehat{T}_f^{\dim Y_\eta+1} \wedge [\mathcal{Y}]\right)\right), \pi^*_{[1]}\phi \right\rangle
\end{aligned}$$

where we used $\pi^*_{[m]}\phi = p_1^*(\pi^*_{[1]}\phi)$. By Fubini, we have

$$(p_1)_* \left(\bigwedge_{j=2}^m p_j^* \left(\widehat{T}_f^{\dim Y_\eta+1} \wedge [\mathcal{Y}]\right)\right) = \pi^*_{[1]} \left((T^{(1)}_{f,\mathcal{Y}})^{m-1}\right)$$

so that

$$\langle T^{(m)}_{f,\mathcal{Y}}, \phi\rangle \geq \left\langle \widehat{T}_f^{\dim Y_\eta+1} \wedge [\mathcal{Y}] \wedge \pi^*_{[1]} \left((T^{(1)}_{f,\mathcal{Y}})^{m-1}\right), \pi^*_{[1]}\phi \right\rangle$$

which gives the first point.

We prove the second point. Assume $T^{(m)}_{f,\mathcal{Y}} \neq 0$. Then, we develop the product

$$\widehat{T}_{f^{[m]}}^{m(\dim Y_\eta+1)} = \left(p_1^*(\widehat{T}_f) + \cdots + p_m^*(\widehat{T}_f)\right)^{m(\dim Y_\eta+1)}$$

in $T^{(m)}_{f,\mathcal{Y}}$. We deduce that there exists $(\alpha_1, \ldots \alpha_m)$ with $\alpha_1 + \cdots + \alpha_m = m(\dim Y_\eta + 1)$ such that

$$\bigwedge_{j=1}^m p_j^* \left(\widehat{T}_f^{\alpha_j} \wedge [\mathcal{Y}]\right) > 0.$$

By symmetry, we can assume that $\alpha_2 + \cdots + \alpha_m \geq (m-1)(\dim Y_\eta + 1)$. Take ϕ a test form on $S^0_{\mathcal{Y}}$ so that

$$0 \neq \left\langle \bigwedge_{j=1}^m p_j^* \left(\widehat{T}_f^{\alpha_j} \wedge [\mathcal{Y}]\right), \pi^*_{[m]}(\theta) \right\rangle = \left\langle \widehat{T}_f^{\alpha_1} \wedge [\mathcal{Y}] \wedge p_1^* \left(\bigwedge_{j=2}^m p_j^* \left(\widehat{T}_f^{\alpha_j} \wedge [\mathcal{Y}]\right)\right), \pi^*_{[1]}(\theta) \right\rangle.$$

In particular, we deduce that $p_1^*\left(\bigwedge_{j=2}^m p_j^*\left(\widehat{T}_f^{\alpha_j}\wedge[\mathcal{Y}]\right)\right)$ is non zero, which in turn implies $p_1^*\left(\bigwedge_{j=2}^m p_j^*\left(\widehat{T}_f^{\alpha'_j}\wedge[\mathcal{Y}]\right)\right)$ where the α'_j are integers such that $\alpha'_j\leq\alpha_j$ for all j and $\alpha'_2+\cdots+\alpha'_m=(m-1)(\dim Y_\eta+1)$. This in turn implies that $T_{f,\mathcal{Y}}^{(m-1)}\neq 0$. The case $m\geq\dim S$ is similar.

The proof of the third point is now similar to that of the first point. Indeed, assume $\dim Y_\eta=\dim X_\eta-1$, then $p_j^*(\widehat{T}_f)^{\dim Y_\eta+2}\wedge p_j^*([\mathcal{Y}])=p_j^*(\widehat{T}_f^{\dim Y_\eta+2}\wedge[\mathcal{Y}])=0$. In particular,

$$\widehat{T}_{f^{[\dim S]}}^{\dim S(\dim Y_\eta+1)}\wedge[\mathcal{Y}^{[\dim S]}]=\prod_{j=1}^{\dim S}\binom{j\dim X_\eta}{\dim X_\eta}\bigwedge_{j=1}^{\dim S}p_j^*\left(\widehat{T}_f)^{\dim Y_\eta+1}\wedge[\mathcal{Y}]\right)$$

and the rest follows. To conclude, take $m\geq\dim S$:

$$\begin{aligned}(\pi_{[m]})_*\left(\widehat{T}_{f^{[m]}}^{\dim\mathcal{Y}^{[m]}}\wedge[\mathcal{Y}^{[m]}]\right)&=(\pi_{[m]})_*\left(\left(p_1^*(\widehat{T}_f)+\cdots+p_m^*(\widehat{T}_f)\right)^{\dim\mathcal{Y}^{[m]}}\wedge[\mathcal{Y}^{[m]}]\right)\\&=(\pi_{[m]})_*\left(\left(p_1^*(\widehat{T}_f)+\cdots+p_m^*(\widehat{T}_f)\right)^{m\dim Y_\eta+\dim S}\wedge[\mathcal{Y}^{[m]}]\right).\end{aligned}$$

Developing all terms in the product and using $\widehat{T}_f^{\dim Y_\eta+2}\wedge[\mathcal{Y}]=0$, we end up, up to permutations, to a sum of terms of the form

$$(\pi_{[m]})_*\left(\bigwedge_{j=1}^{\dim S}\left(p_j^*(\widehat{T}_f^{\dim Y_\eta+1}\wedge[\mathcal{Y}]\right)\wedge\bigwedge_{j=\dim S+1}^{m}\left(p_j^*(\widehat{T}_f^{\dim Y_\eta}\wedge[\mathcal{Y}])\right)\right).$$

The assertion now follows by Fubini. □

1.3. **The dynamical volumes of a pair.** As above, let $(\mathcal{X},f,\mathcal{L},\mathcal{Y})$ be a dynamical pair parametrized by S with regular part $S^0_{\mathcal{Y}}$. We now can define the dynamical volumes of $\mathcal{Y}$ as follows

Definition 1.5. *For any $m\geq\dim S$, we define the m-*dynamical volume $\mathrm{Vol}_f^{(m)}(\mathcal{Y})$ *of $\mathcal{Y}$ for $(\mathcal{X},f,\mathcal{L})$ as the non-negative real number*

$$\mathrm{Vol}_f^{(m)}(\mathcal{Y}):=\int_{(\mathcal{X}^{[m]})^0(\mathbb{C})}\widehat{T}_{f^{[m]}}^{\dim\mathcal{Y}^{[m]}}\wedge[\mathcal{Y}^{[m]}].$$

*For any ample $\mathbb{Q}$-line bundle $\mathcal{M}$ on S, we also define the m-*parametric degree $\deg_{f,\mathcal{M}}^{(m)}(\mathcal{Y})$ *of $\mathcal{Y}$ relative to $\mathcal{M}$ as*

$$\deg_{f,\mathcal{M}}^{(m)}(\mathcal{Y}):=\int_{(\mathcal{X}^{[m]})^0(\mathbb{C})}\widehat{T}_{f^{[m]}}^{\dim\mathcal{Y}^{[m]}-1}\wedge[\mathcal{Y}^{[m]}]\wedge\pi_{[m]}^*\left(\omega_S\right),$$

where ω_S is any smooth form on S representing $c_1(\mathcal{M})$.

Remark 1.6. When $\dim S=1$, an easy computation gives

$$\deg_{f,\mathcal{M}}^{(1)}(\mathcal{Y})=\deg_{Y_\eta}(L_\eta)\cdot\deg_{\mathcal{M}}(S)>0.$$

In particular, if $\deg_{\mathcal{M}}(S)=1$, then

$$\widehat{h}_{f_\eta}(Y_\eta)=\frac{\mathrm{Vol}_f^{(1)}(\mathcal{Y})}{(\dim Y_\eta+1)\cdot\deg_{f,\mathcal{M}}^{(1)}(\mathcal{Y})}.$$

We can relate the non-vanishing of the bifurcation measure $\mu_{f,\mathcal{Y}}$ with the non vanishing of the volumes $\mathrm{Vol}_f^{(m)}(\mathcal{Y})$ for all $m \geq \dim S$.

Proposition 1.7. *The following properties hold:*

(1) We have $\mu_{f,\mathcal{Y}}$ is non-zero if and only if for all $m \geq \dim S$, $\mathrm{Vol}_f^{(m)}(\mathcal{Y}) > 0$.

(2) For any integer $m \geq \dim S$, and for any ample $\mathbb{Q}$-line bundle $\mathcal{M}$ on S of volume 1, we have

$$\deg_{f,\mathcal{M}}^{(m)}(\mathcal{Y}) \geq m \deg_{Y_\eta}(L_\eta)^{m-\dim S+1} \int_{S^0(\mathbb{C})} T_{f,\mathcal{Y}}^{(\dim S-1)} \wedge \omega_S,$$

for any smooth form ω_S which represents $c_1(\mathcal{M})$.

In particular, if $\mu_{f,\mathcal{Y}} \neq 0$, then for all $m \geq \dim S$, and all $\mathcal{M}$, we have $\mathrm{Vol}_f^{(m)}(\mathcal{Y}) > 0$ and $\deg_{f,\mathcal{M}}^{(m)}(\mathcal{Y}) > 0$.

Proof. The first point follows from Proposition 1.4. Let $p := \dim S$. To prove the second point, we remark that

$$\widehat{T}_{f^{[m]}}^{\dim \mathcal{Y}^{[m]}-1} \geq \left(\bigwedge_{j=p}^{m} p_{\ell_j}^* \widehat{T}_f^{\dim Y_\eta} \right) \wedge \bigwedge_{j=1}^{p-1} \left(p_j^* \widehat{T}_f^{\dim Y_\eta+1} \right).$$

Let $\pi_p : \mathcal{X}^{[m]} \to \mathcal{X}^{[p-1]}$ be the projection forgetting the $m-p+1$ last variables and for $1 \leq j \leq p-1$, let $p_j' : \mathcal{X}^{[p-1]} \to \mathcal{X}$ be the projection onto the j-th factor. The measure

$$\left(\bigwedge_{j=p}^{m} p_{\ell_j}^* \widehat{T}_f^{\dim Y_\eta} \right) \wedge \bigwedge_{j=1}^{p-1} \left(p_j^* \widehat{T}_f^{\dim Y_\eta+1} \right) \wedge [\mathcal{Y}^{[m]}] \wedge \pi_{[m]}^*(\omega_S)$$

rewrites as

$$\bigwedge_{\ell=p}^{m} p_\ell^* \left(\widehat{T}_f^{\dim Y_\eta} \wedge [\mathcal{Y}] \right) \wedge (\pi_p)^* \left(\bigwedge_{j=1}^{p-1} (p_j')^* \left(\widehat{T}_f^{\dim Y_\eta+1} \wedge [\mathcal{Y}] \right) \wedge \pi_{[p-1]}^*(\omega_S) \right)$$

where we used that $\pi_{[p]}^* \omega_S = (\pi_p)^* \left(\pi_{[p-1]}^* \omega_S \right)$. In particular, its volume is that of its push-forward by π_p, which is the measure

$$(\pi_p)_* \left(\bigwedge_{\ell=p}^{m} p_\ell^* \left(\widehat{T}_f^{\dim Y_\eta} \wedge [\mathcal{Y}] \right) \right) \wedge \bigwedge_{j=1}^{p-1} (p_j')^* \left(\widehat{T}_f^{\dim Y_\eta+1} \wedge [\mathcal{Y}] \right) \wedge \pi_{[p-1]}^*(\omega_S).$$

We now remark that π_p has fibers of dimension $k(m-p+1) := \dim X_\eta^{m-p+1}$ and that $\widehat{T}_f^{\dim Y_\eta} \wedge [\mathcal{Y}]$ is a (k,k)-current on $\mathcal{X}^0(\mathbb{C})$, so that the current

$$T := (\pi_p)_* \left(\bigwedge_{\ell=p}^{m} p_\ell^* \left(\widehat{T}_f^{\dim Y_\eta} \wedge [\mathcal{Y}] \right) \right)$$

is a $(0,0)$-current on $(\mathcal{X}^{[p-1]})^0(\mathbb{C})$ which is nothing but the constant $\deg_{Y_\eta}(L_\eta)^{m-p+1}$. Therefore, the volume of the studied measure is exactly

$$\deg_{Y_\eta}(L_\eta)^{m-p+1} \cdot \int_{S^0(\mathbb{C})} T_{f,\mathcal{Y}}^{(\dim S-1)} \wedge \omega_S.$$

As the wedge product is symmetric, proceeding similarly for the other terms of the sum, we find

$$\deg_{f,\mathcal{M}}(\mathcal{Y}) \geq m \cdot \deg_{Y_\eta}(L_\eta)^{m-p+1} \cdot \int_{S^0(\mathbb{C})} T_{f,\mathcal{Y}}^{(\dim S-1)} \wedge \omega_S,$$

and the proof of the second point is complete (observe that the second point of Proposition 1.4 guarantees that $T_{f,\mathcal{Y}}^{(\dim S-1)} \neq 0$). □

1.4. **Dynamical volume as limits of iterated intersection numbers.** Let $(\mathcal{X}, f, \mathcal{L})$ be a family of polarized endomorphisms of degree d and $\mathcal{Y} \subsetneq \mathcal{X}$ be a subvariety with $\pi(\mathcal{Y}) = S$. Let also $m \geq 1$ be an integer and let $(\mathcal{X}^{[m]}, f^{[m]}, \mathcal{L}^{[m]})$ be the polarized endomorphism induced on $\mathcal{X}^{[m]} := \mathcal{X} \times_S \cdots \times_S \mathcal{X}$ as above with induced morphism $\pi_{[m]} : \mathcal{X}^{[m]} \to S$ and let $\mathcal{Y}^{[m]} := \mathcal{Y} \times_S \cdots \times_S \mathcal{Y}$. One can check that $\pi_{[m]}(\mathcal{Y}^{[m]}) = S$ and we have the following which is essentially immediate.

Lemma 1.8. *For any $m \geq 1$, there is a sequence $(\mathcal{X}_n^{(m)})_{n\geq 0}$ of projective varieties, a sequence $\psi_n^{(m)} : \mathcal{X}_n^{(m)} \to \mathcal{X}^{[m]}$ of birational morphisms which are isomorphisms above the regular part of $(\mathcal{X}^{[m]})^0$ and a sequence of morphisms $F_n^{(m)} : \mathcal{X}_n^{(m)} \to \mathcal{X}^{[m]}$ such that $\mathcal{X}_0^{(m)} = \mathcal{X}^{[m]}$ and the following diagram commutes*

$$\begin{tikzcd} \mathcal{X}_n^{(m)} \arrow[d, "\psi_n^{(m)}"'] \arrow[dr, "F_n^{(m)}"] & \\ \mathcal{X}^{[m]} \arrow[r, dashed, "(f^{[m]})^n"'] & \mathcal{X}^{[m]} \end{tikzcd} .$$

Moreover, one can choose $\mathcal{X}_n^{(m)}$ as a finite sequence of blow-ups of $\mathcal{X}_{n-1}^{(m)}$.

Relying on estimates from [GV] we can deduce

Lemma 1.9. *For any $m \geq \dim S$, there is a constant $C_m \geq 1$ depending only on $(\mathcal{X}, f, \mathcal{L}, \mathcal{Y})$ and m such that for any $n \geq 1$, if $\mathcal{Y}_n^{(m)} := (F_n^{(m)})_*(\psi_n^{(m)})^*\mathcal{Y}^{[m]}$, then*

$$\left| \frac{\left(\{\mathcal{Y}_n^{(m)}\} \cdot c_1(\mathcal{L}^{[m]})^{\dim \mathcal{Y}^{[m]}}\right)}{d^{n \dim \mathcal{Y}^{[m]}}} - \mathrm{Vol}_f^{(m)}(\mathcal{Y}) \right| \leq C_m d^{-n},$$

and, for any ample $\mathbb{Q}$-line bundle $\mathcal{M}$ on S of volume 1,

$$\left| \frac{\left(\{\mathcal{Y}_n^{(m)}\} \cdot c_1(\mathcal{L}^{[m]})^{\dim \mathcal{Y}^{[m]}-1} \cdot c_1(\pi_{[m]}^*\mathcal{M})\right)}{d^{n(\dim \mathcal{Y}^{[m]}-1)}} - \deg_{f,\mathcal{M}}^{(m)}(\mathcal{Y}) \right| \leq C_m d^{-n}.$$

Proof. Let ω_S be a smooth form on $S(\mathbb{C})$ which represents $c_1(\mathcal{M})$ (it has mass $1 = \deg_S(\mathcal{M})$) and $\omega_{\mathcal{L}}$ be a smooth form on $\mathcal{X}(\mathbb{C})$ which represents $c_1(\mathcal{L})$. For $m \geq \dim S$, define $\omega^{[m]} := \sum_j p_j^*\omega$. Let $(\mathcal{X}^{[m]})^0 := \pi_{[m]}^{-1}(S_{\mathcal{Y}}^0)$ as above. By definition, we have

$$\begin{aligned} \left(\{\mathcal{Y}_n^{(m)}\} \cdot c_1(\mathcal{L}^{[m]})^{\dim \mathcal{Y}^{[m]}}\right) &= \int_{(\mathcal{X}^{[m]})^0(\mathbb{C})} \left(\omega^{[m]}\right)^{\dim \mathcal{Y}^{[m]}} \wedge [\mathcal{Y}_n^{(m)}] \\ &= \int_{(\mathcal{X}^{[m]})^0(\mathbb{C})} \left((f^{[m]})^n)^*\omega^{[m]}\right)^{\dim \mathcal{Y}^{[m]}} \wedge [\mathcal{Y}^{[m]}]. \end{aligned}$$

We rely on Proposition 3.4 of [GV]: we have

$$d^{-n\dim \mathcal{Y}^{[m]}}\left(\{\mathcal{Y}_n^{(m)}\}\cdot c_1(\mathcal{L}^{[m]})^{\dim \mathcal{Y}^{[m]}}\right)=\int_{(\mathcal{X}^{[m]})^0(\mathbb{C})}\widehat{T}_{f^{[m]}}^{\dim \mathcal{Y}^{[m]}}\wedge[\mathcal{Y}^{[m]}]+O\left(\frac{1}{d^n}\right).$$

This is the first assertion we want to prove. Similarly,

$$\begin{aligned} I_{n,m}:&=\left(\{\mathcal{Y}_n^{(m)}\}\cdot c_1(\mathcal{L}^{[m]})^{\dim \mathcal{Y}^{[m]}-1}\cdot c_1(\pi_{[m]}^*\mathcal{M})\right)\\ &=\int_{(\mathcal{X}^{[m]})^0(\mathbb{C})}\left(((f^{[m]})^n)^*\omega^{[m]}\right)^{\dim \mathcal{Y}^{[m]}-1}\wedge[\mathcal{Y}^{[m]}]\wedge(\pi_{[m]}^*\omega_S)\end{aligned}$$

and the same argument using Proposition 3.4 of [GV] gives

$$d^{-n(\dim \mathcal{Y}^{[m]}-1)}I_{n,m}=\int_{(\mathcal{X}^{[m]})^0(\mathbb{C})}\widehat{T}_{f^{[m]}}^{\dim \mathcal{Y}^{[m]}-1}\wedge[\mathcal{Y}^{[m]}]\wedge\pi_{[m]}^*(\omega_S)+O\left(\frac{1}{d^n}\right).$$

This concludes the proof. □

1.5. **A sufficient criterion for positive volume.** To finish this section, we would like to give a sufficient criterion for a parameter to belong to the support of the measure $\mu_{f,\mathcal{Y}}$. The existence of such a parameter implies in particular that $\mathrm{Vol}_f(\mathcal{Y})>0$.

Definition 1.10. *Pick an integer $m\geq 1$. We say that $\mathcal{Y}$ is m-*transversely J_k-prerepelling (resp. properly J_k-prerepelling) *at a point $z=(z_1,\dots,z_m)\in\mathcal{X}^{[m]}$ with $\lambda_0:=\pi_{[m]}(z)\in S^0$ if $z_1,\dots,z_m$ are $J_k(f_{\lambda_0})$-repelling periodic points of f_{λ_0} and if there exist an integer $N\geq 1$ and a neighborhood U of λ_0 such that, if $z_j(\lambda)$ is the natural continuation of z_j as a repelling periodic point of f_λ in U, then*

(1) $z_j\in f_{\lambda_0}^N(Y_{\lambda_0})$ for all $1\leq j\leq m$,
(2) $z_j(\lambda)\in J_k(f_\lambda)$ for all $\lambda\in U$ and all $1\leq j\leq m$,
(3) the image of the local section $Z:\lambda\in U\mapsto(z_1(\lambda),\dots,z_m(\lambda))\in(\mathcal{X}^{[m]})^0$ of $\pi_{[m]}$ intersects transversely a local branch of $(f^{[m]})^N(\mathcal{Y}^{[m]})$ at z (resp. z lies in an proper intersection between the image of Z and a local branch of $(f^{[m]})^N(\mathcal{Y}^{[m]})$ of pure dimension $\dim S-m$).

In some sense, this definition is equivalent to the existence of m *independent* Misiurewicz intersections. The case of single Misiurewicz intersections corresponds to *Misiurewicz parameters* in [BBD]. The third point in the definition seems a bit technical but in the examples we will construct, we cannot a priori exclude the case where $\mathcal{Y}^{[m]}$ is not locally irreducible and the periodic points lie persistently in a local branch of $\mathcal{Y}$ but transversely to another local branch. Another important remark for what follows is that, as observed by Dujardin (see [Du, Proposition-Definition 2.5]), the repelling periodic points can be replaced by points in a repelling hyperbolic set contained in J_k. Finally, notice that when $m=\dim S$ and $\mathcal{Y}$ is locally irreducible near $z_1,\dots,z_m$, Definition 1.10 is exactly what DeMarco and Mavraki [DM] call a rigid m-repeller.

We prove the following, which is a general criterion in the spirit of [DM, Proposition 4.8].

Proposition 1.11. *Let $(\mathcal{X},f,\mathcal{L})$ be a family of polarized endomorphisms parametrized by S and let $\mathcal{Y}\subsetneq\mathcal{X}$ be a hypersurface which projects dominantly to S. Let $1\leq m\leq\dim S$ and assume $\mathcal{Y}$ is m-properly J_k-prerepelling at $z\in(\mathcal{X}^{[m]})^0$. Then*

$$z\in\operatorname{supp}\left(\widehat{T}_{f^{[m]}}^{m(\dim Y_\eta+1)}\wedge[\mathcal{Y}^{[m]}]\right).$$

In particular, $\pi_{[m]}(z) \in \mathrm{supp}(T^{(m)}_{f,\mathcal{Y}})$.

The proof of this result is an adaptation of the strategy of Buff and Epstein [BE2] and the strategy of Berteloot, Bianchi and Dupont [BBD], see also [Ga2, AGMV, Ga1, GV].

Proof of Proposition 1.11. As the statement is purely local, we let $\mathbb{B} \subset S^0$ be a ball in a local coordinate centered at λ_0. Since $\widehat{T}_f$ has continuous potentials, for any analytic submanifold $\Lambda \subset \mathbb{B}$ of dimension m with $\lambda_0 \in \Lambda$, we have

$$\mathrm{supp}\left(\widehat{T}^{m(\dim Y_\eta+1)}_{f^{[m]}}|_{\pi^{-1}_{[m]}(\Lambda)} \wedge [\mathcal{Y}^{[m]} \cap \pi^{-1}_{[m]}(\Lambda)]\right) \subset \mathrm{supp}\left(\widehat{T}^{m(\dim Y_\eta+1)}_{f^{[m]}} \wedge [\mathcal{Y}^{[m]}]\right)$$

by e.g. [Ga2, Lemma 6.3]. In particular, we can replace $\mathbb{B}$ with the intersection between $\mathbb{B}$ with a subspace $\mathbb{B} \cap V$ where V is a linear subspace of dimension m such that the intersection between the image of the local section $Z : \lambda \in U \cap V \mapsto (z_1(\lambda), \dots, z_m(\lambda)) \in (\mathcal{X}^{[m]})^0$ of $\pi_{[m]}$ and the a local branch of $(f^{[m]})^N(\mathcal{Y}^{[m]})$ at z is isolated in $(\mathcal{X}^{[m]})^0 \cap \pi^{-1}_{[m]}(\mathbb{B} \cap V)$. In the rest of the proof, we thus can assume $m = \dim S$ and let k be the relative dimension of $\mathcal{X}$ over S so that $\dim Y_\eta + 1 = k$. To simplify notations, write $F := f^{[m]} : (\mathcal{X}^{[m]})^0 \to (\mathcal{X}^{[m]})^0$ and we let $\mu := T^{(m)}_{f,\mathcal{Y}}|_{\mathbb{B}}$.

Our aim here is to exhibit a basis of neighborhood $\{\Omega_n\}_n$ of λ_0 in $\mathbb{B}$ with $\mu(\Omega_n) > 0$ for all n. For a Borel subset $B \subset \mathbb{B}$, let $(\mathcal{X}^{[m]})_B := \pi^{-1}_{[m]}(B)$, where $\pi_{[m]} : \mathcal{X}^{[m]} \to S$ is the map induced by $\pi : \mathcal{X} \to S$. Then, since $F^*\widehat{T}_F = d\widehat{T}_F$, we have

$$(F^n)^*\left(\widehat{T}^{km}_F\right) = d^{mkn}\widehat{T}^{km}_F \quad \text{and} \quad \mu(B) \;=\; d^{-kmn}\int_{(\mathcal{X}^{[m]})_B} \widehat{T}^{mk}_F \wedge (F^n)_*\left[\mathcal{Y}^{[m]}\right].$$

Since $\mathcal{Y}$ is properly J_k-prerepelling at λ_0, there are $z_1, \dots, z_m \in X_{\lambda_0}$, $J_k(f_{\lambda_0})$-repelling periodic points and $N \geq 1$ such that $(z_1, \dots, z_m) \in F^N(\mathcal{Y}^{[m]})^0$. Let $p \geq 1$ be such that $f^p_{\lambda_0}(z_i) = z_i$ for all i. We let $\mathcal{Y}_0$ be the local branch of $(F^N)(\mathcal{Y}^{[m]})^0$ satisfying the hypothesis of the Proposition. For any integer $n \geq 1$, we let $\mathcal{Y}_n := (F^{np})(\mathcal{Y}_0)$, so that $\dim(\mathcal{Y}_n) = mk$ and

$$I_n := \int_{(\mathcal{X}^{[m]})_{\mathbb{B}}} \widehat{T}^{mk}_F \wedge [\mathcal{Y}_n] \leq d^{knm}\mu(\mathbb{B}).$$

Up to reducing $\mathbb{B}$, we can assume $z_i(\lambda)$ is $J_k(f_\lambda)$-repelling for all $\lambda \in \mathbb{B}$ and that there is $K > 1$ such that

$$d(f^p_\lambda(z), f^p_\lambda(w)) \geq K \cdot d(z, w)$$

for all $z, w \in B(z_j, \epsilon) \subset \mathcal{X}$ and all $\lambda \in \mathbb{B}$ for some given $\epsilon > 0$ with $\pi(B(z_j, \epsilon)) \subset \mathbb{B}$ [BBD]. Thus, if we denote $\mathrm{z} := (z_1, \dots, z_m) \in \mathcal{X}^{[m]}$ and by S_n the connected component of $\mathcal{Y}_n \cap B_\epsilon$ where $B_\epsilon := B_{\mathcal{X}^{[m]}}(\mathrm{z}, \epsilon)$, the current $[S_n]$ is vertical-like in B_ϵ and there exist $n_0 \geq 1$ and a basis of neighborhood Ω_n of λ_0 in $\mathbb{B}$ such that for all $n \geq n_0$

$$\mathrm{supp}([S_n]) = S_n \subset \mathcal{X}^{[m]}_{\Omega_n} \cap B_\epsilon.$$

Let S be any weak limit of the sequence $[S_n]/\|[S_n]\|$, where the mass $\|[S_n]\|$ is computed with respect to some Kähler form α on $\mathcal{X}^{[m]}_{\mathbb{B}}$. Then S is a closed positive (k,k)-current of mass 1 in B_ϵ whose support is contained in the fiber $X^m_{\lambda_0}$ of $\pi_{[m]}$. Hence $S = M \cdot [X^m_{\lambda_0} \cap B_\epsilon]$, where $M^{-1} > 0$ is the volume of B_ϵ for the volume form $\alpha|_{X^m_{\lambda_0}}$.

As a consequence, $[S_n]/\|[S_n]\|$ converges weakly to S as $n \to \infty$ and, since the (mk, mk)-current $\widehat{T}_F^{km}$ is the mk-times wedge product of a closed positive $(1,1)$-current with continuous potential,

$$\widehat{T}_F^{km} \wedge \frac{[S_n]}{\|[S_n]\|} \longrightarrow \widehat{T}_F^{km} \wedge S$$

as $n \to +\infty$. Whence the above gives

$$\begin{aligned}\liminf_{n\to\infty} \left(\|[S_n]\|^{-1} \cdot I_n\right) &\geq \liminf_{n\to\infty} \int \widehat{T}_F^{km} \wedge \frac{[S_n]}{\|[S_n]\|} \geq \int \widehat{T}_F^{km} \wedge S \\ &\geq \ M \cdot \int \widehat{T}_F^{km} \wedge [X_{\lambda_0}^m \cap B_\epsilon].\end{aligned}$$

In particular, there exists $n_1 \geq n_0$ such that for all $n \geq n_1$,

$$2\|[S_n]\|^{-1} \cdot I_n \geq M \int \widehat{T}_F^{km} \wedge [X_{\lambda_0}^m \cap B_\epsilon].$$

Finally, by construction of S_n, we have $\liminf_{n\to\infty} \|[S_n]\| \geq \mathrm{Vol}(B_\varepsilon) > 0$, where the volume is computed with respect to the Kähler form $\alpha|_{X_{\lambda_0}^m}$ on $X_{\lambda_0}^m$. Up to increasing n_0, we may assume $\|[S_n]\| \geq c > 0$ for all $n \geq n_0$. Letting $\gamma = Mc/4 > 0$, we find

$$\mu(\mathbb{B}) = d^{-km(np+N)} I_n \geq d^{-km(np+N)} \gamma \int \widehat{T}_F^{km} \wedge [X_{\lambda_0}^m \cap B_\epsilon],$$

for all $n \geq n_1$. To conclude, we need to prove the last integral is non-zero. By construction, the set $X_{\lambda_0}^m \cap B_\epsilon$ is an open neighborhood of z in $X_{\lambda_0}^m$ whence it contains $B(z_1, \delta) \times \cdots \times B(z_m, \delta) \subset X_{\lambda_0}^m$ for some $\delta > 0$ (with a slight abuse of notations since here the balls are meant in X_{λ_0}). Moreover, the current $\widehat{T}_F^{km}$ restricts to $X_{\lambda_0}^{km}$ as the measure

$$\widehat{T}_F^{km}|_{X_{\lambda_0}^m} = \mu_{F_{\lambda_0}} = \mu_{f_{\lambda_0}}^{\otimes m}.$$

In particular, we can apply Fubini Theorem to find

$$\int \widehat{T}_F^{km} \wedge [X_{\lambda_0}^m \cap B_\epsilon] \geq \int_{X_{\lambda_0}} \widehat{T}_F^{km} \wedge [B(z_1, \delta) \times \cdots \times B(z_m, \delta)] = \prod_{j=1}^m \mu_{f_{\lambda_0}}(B(z_j, \delta)) > 0,$$

where we used that $z_j \in \mathrm{supp}(\mu_{f_{\lambda_0}})$ by assumption. □

2. Rigidity of some stable families

2.1. **Spaces of endomorphisms, moduli spaces, stable families.**

2.1.1. *The spaces* End_d^k *and* Poly_d^k. As an endomorphism f of $\mathbb{P}^k$ of degree d is given by $k+1$ homogeneous polynomials of degree d, the coefficients of these polynomials allow us to see f as a point in $\mathbb{P}^{N_d^k}$ where $N_d^k := (k+1)\binom{k+d}{d} - 1$. The condition on the coefficients to ensure that the associated map is an endomorphism of $\mathbb{P}^k$ is algebraic so there exists a Zariski open set $\mathrm{End}_d^k \subset \mathbb{P}^{N_d^k}$ corresponding to degree d endomorphisms. More precisely, the variety End_d^k is the complement of the hypersurface in $\mathbb{P}^{N_d^k}$ defined by the vanishing of the Macaulay resultant (see, e.g., [BB1] for details). In particular, it is an irreducible smooth quasi-projective variety defined over $\mathbb{Q}$. In particular, the map

$$f : \mathbb{P}^k_{\mathrm{End}_d^k} \longrightarrow \mathbb{P}^k_{\mathrm{End}_d^k}$$

is a family $(\mathbb{P}^k_{\mathbb{P}^{N_d^k}}, f, \mathcal{O}_{\mathbb{P}^k}(1))$ of degree d endomorphisms of $\mathbb{P}^k$ parametrized by $\mathbb{P}^{N_d^k}$ – if we follow the notations introduced above – which is defined over $\mathbb{Q}$.

A *regular polynomial endomorphism* $f : \mathbb{A}^k \to \mathbb{A}^k$ of degree $d \geq 2$ is a polynomial map which extends to a degree d endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$. For such a morphism, if H_∞ is the hyperplane at infinity of $\mathbb{A}^k$ in $\mathbb{P}^k$, we have $f^{-1}(H_\infty) = H_\infty$, , see, e.g., [BJ]. The space Poly_d^k of regular polynomial endomorphisms of degree d of $\mathbb{A}^k$ is a smooth closed subvariety of End_d^k of dimension $k\binom{k+d}{d}$ – which is the intersection of End_d^k with a linear subspace of $\mathbb{P}^{N_d^k}$ defined over $\mathbb{Q}$. In particular, Poly_d^k is also a smooth quasi-projective variety defined over $\mathbb{Q}$ and the map

$$f : \mathbb{P}^k_{\mathrm{Poly}_d^k} \longrightarrow \mathbb{P}^k_{\mathrm{Poly}_d^k}$$

is a family $(\mathbb{P}^k_S, f, \mathcal{O}_{\mathbb{P}^k}(1))$ of degree d endomorphisms of $\mathbb{P}^k$ parametrized by the closure S of Poly_d in $\mathbb{P}^{N_d^k}$ – if we follow the notations introduced above – which is defined over $\mathbb{Q}$.

2.1.2. *The moduli spaces $\mathscr{M}_d^k$ and $\mathscr{P}_d^k$.* The first family we want to investigate is the moduli space $\mathscr{M}_d^k$ of degree d endomorphisms of the projective space $\mathbb{P}^k$ of dimension k: it is the quotient space of the space End_d^k of endomorphisms of degree d of $\mathbb{P}^k$ by the action by conjugacy of $PGL(k+1)$. It is known to be an irreducible affine variety of dimension

$$\mathcal{N}_d^k := \dim \mathscr{M}_d^k = (k+1)\binom{k+d}{d} - (k+1)^2$$

defined over $\mathbb{Q}$, see [Si2] when $k = 1$ and [PST] when $k > 1$, whence there is a proper closed subvariety V such that the canonical projection

$$\Pi : \mathrm{End}_d^k \setminus V \to \mathscr{M}_d^k \setminus \Pi(V)$$

is an $PGL(k+1)$-principal bundle. We consider the universal family

$$f : \mathbb{P}^k_{\mathscr{M}_d^k} \longrightarrow \mathbb{P}^k_{\mathscr{M}_d^k}$$

which is a family $(\mathbb{P}^k_S, f, \mathcal{O}_{\mathbb{P}^k}(1))$ of degree d endomorphisms of $\mathbb{P}^k$ parametrized by a projective model S of $\mathscr{M}_d^k$ with regular part $\mathscr{M}_d^k$ – if we follow the notations introduced above – which is defined over $\mathbb{Q}$.

The second family we investigated is the moduli space $\mathscr{P}_d^k$ of degree d regular polynomial endomorphisms of the affine space $\mathbb{A}^k$: it is the quotient of the space Poly_d^k of regular polynomial endomorphisms of degree d of the affine space $\mathbb{A}^k$ by the action by conjugacy of $\mathrm{Aut}(\mathbb{A}^k) = GL(k) \ltimes \mathbb{A}^k$. The same proof as those given in [Si1, PST] ensures that the moduli space $\mathscr{P}_d^k$ is also a fine moduli space and is an irreducible affine variety defined over $\mathbb{Q}$ of dimension

$$\mathcal{P}_d^k := \dim \mathscr{P}_d^k = k\binom{k+d}{d} - (k^2+k) > \mathcal{N}_d^{k-1} = \dim \mathscr{M}_d^{k-1}.$$

We also consider the universal family

$$f : \mathbb{P}^k_{\mathscr{P}_d^k} \longrightarrow \mathbb{P}^k_{\mathscr{P}_d^k}$$

and as above, this is a family $(\mathbb{P}^k_S, f, \mathcal{O}_{\mathbb{P}^k}(1))$ of degree d endomorphisms of $\mathbb{P}^k$ parametrized by a projective model S of $\mathscr{P}_d^k$ with regular part $\mathscr{P}_d^k$ which is defined over $\mathbb{Q}$. As before, there is a proper closed subvariety V such that the canonical projection

$$\Pi : \mathrm{Poly}_d^k \setminus V \to \mathscr{P}_d^k \setminus \Pi(V)$$

is an $\mathrm{Aut}(\mathbb{A}^k)$-principal bundle.

2.1.3. *Stable families of endomorphisms of $\mathbb{P}^k$ following Berteloot-Bianchi-Dupont.* Let M be a connected complex manifold. An analytic family of endomorphisms of $\mathbb{P}^k$ parametrized by M can be described as a surjective holomorphic map $f : (z,t) \in \mathbb{P}^k \times M \longmapsto (f_t(z), t) \in \mathbb{P}^k \times M$. In particular, for any $t \in M$, the induced map $f_t : \mathbb{P}^k \to \mathbb{P}^k$ is an endomorphism of degree d (independent of $t \in M$).

Following Berteloot, Bianchi and Dupont [BBD], we say that such an analytic family of endomorphisms of $\mathbb{P}^k$ is J_k-*stable* if the function

$$t \in M \longmapsto L(f_t) := \int_{\mathbb{P}^k(\mathbb{C})} \log|\det(Df_t)|\mu_{f_t},$$

is a pluriharmonic function on M, i.e. $dd^c_t L(f_t) \equiv 0$, where $L(f_t)$ is the sum of Lyapunov exponents of the unique maximal entropy measure μ_{f_t} of f_t. Berteloot-Bianchi-Dupont gave several equivalent description of this notion of stability and showed it is the higher-dimensional equivalent to the notion of stability introduced by Mañé, Sad and Sullivan [MSS] for families of rational maps of $\mathbb{P}^1$.

When M is a quasi-projective variety, and f is a morphism and, if S is a projective model of M, $(\mathbb{P}^k \times M, f, \mathcal{O}_{\mathbb{P}^k}(1))$ is a family of endomorphisms as above with regular part M. In this case, one can show that the family M is J_k-stable if and only if the function $t \mapsto L(f_t)$ is constant on M and, by [BB1], one also has

$$dd^c L = \pi_* \left(\widehat{T}^k_f \wedge [\mathrm{Crit}(f)] \right) = T_{f,\mathrm{Crit}},$$

as currents on M, so that the family is J_k-stable if and only if $T_{f,\mathrm{Crit}(f)} = 0$ on M. Here $\mathrm{Crit}(f) = \{(z,t) \in \mathbb{P}^k \times M,\ \det(Df_t)(z) = 0\}$.

2.2. **Families with many constant multipliers.** Let $d \geq 2$ and S be an irreducible complex projective variety. Let $(\mathbb{P}^k_S, f, \mathcal{O}_{\mathbb{P}^k}(1))$ be a family of endomorphisms of $\mathbb{P}^k$ of degree d, parametrized by a Zariski open subset S^0 of S. Let $t_0 \in S^0$ be an arbitrary parameter in this family. We consider a non-decreasing sequence $(m_n)_{n\geq 1}$ of positive integers and a sequence of points $(x_n)_{n\geq 1}$ in $\mathbb{P}^k$ such that

- for each $n \geq 1$, x_n is a repelling periodic point for f_{t_0} of exact period m_n,
- no two such point x_n and $x_{n'}$ belong to the same orbit of f_{t_0},
- if for $s \geq 1$ we set $M_s := \{n \geq 1\,;\, m_n \leq s\}$ then $M_s/(sd^{sk})$ converges to 1 when s goes to $\infty/$

In other words, the last point says that most of the periodic cycles of f_{t_0} are orbits of points in $(x_n)_{n\geq 1}$. From these data, for each $n \geq 1$ we consider the analytic set

$$\tilde{X}_n := \left\{ (t, z_1, \ldots, z_n) \in S^0 \times (\mathbb{P}^k)^n\,;\ f_t^{m_s}(z_s) = z_s \text{ for all } 1 \leq s \leq n \right\}.$$

Observe that, since the points in the sequence $(x_n)_{n\geq 1}$ are repelling, the point $(t_0, x_1, \ldots, x_n)$ is regular in $\tilde{X}_n$ and we denote by X_n the irreducible component of $\tilde{X}_n$ which contains it. The natural projection $\pi_n \colon X_n \to S^0$ is surjective and finite. We also have a family of *multiplier maps* $\Lambda_n \colon X_n \to \mathbb{C}^n$ defined by $\Lambda_n(t, z_1, \ldots, z_n) = (\det D_{z_s} f_t^{m_s})_{1\leq s\leq n}$

Proposition 2.1. *Assume that there exists $t_1 \in S^0$ such that there is no algebraic curve $Z \subset S^0$ passing through t_1 such that Z is J_k-stable. Then for $n \geq 1$ large enough the multiplier map Λ_n is generically finite-to-one.*

Proof. Observe first that, the maps Λ_n contain more and more information, if the result holds for one $n_0 \geq 1$ then it is also the case for all $n \geq n_0$. Assume by contradiction that for each $n \geq 1$ the map Λ_n is not generically finite. In particular, for each $n \geq 1$ the set $Y_n = \Lambda_n^{-1}(\Lambda_n(t_1))$ has positive dimension. The sequence of algebraic set $(Z_n)_{n\geq 1}$ defined by $Z_n := \pi_n(Y_n)$ is decreasing so there exists $N \geq 1$ such that $Z_n = Z_N$ for all $n \geq N$. From this, the key observation is that, relying on the equidistribution of repelling orbits [BD6], we have by [BDM, Theorem 1.5] (see also [BD2, Theorem 4.1]) for all $t \in S^0$,

$$\lim_{n\to+\infty} \frac{1}{d^{kn}} \sum_{p\in \mathrm{RPer}_n(f_t)} \log|\det(Df_t)(p)| = L(f)$$

where $\mathrm{RPer}_n(f_t)$ is the set of n-periodic repelling points of f_t and $L(f_t)$ the sum of the Lyapunov exponents of its equilibrium measure. This implies by the chain rules that

$$\lim_{n\to+\infty} \frac{1}{nd^{kn}} \sum_{p\in \mathrm{RPer}_n(f_t)} \log|\det(Df_t^n)(p)| = L(f_t)$$

or equivalently

$$\lim_{n\to+\infty} \frac{1}{nd^{kn}} \sum_{p\in \mathrm{Per}_n(f_t)} \log^+|\det(Df_t^n)(p)| = L(f_t)$$

where $\log^+ x = \max(\log x, 0)$ and $\mathrm{Per}_n(f_t)$ is the set of all n-periodic points of f_t. In particular, as we have assume that $M_s/(sd^{sk}) \to 1$ with s where $M_s := \{n \geq 1\,;\, m_n \leq s\}$, the fact that all the functions Λ_n are constant on Y_n implies that $t \mapsto L(f_t)$ is also constant on Z_N. In particular Z_N is a J_k-stable family containing t_1. Contradiction. □

Using isolated Lattès maps, we easily have the following result which answers by the positive to the first part of [DS5, Question 19.4], a description of the set Γ in the next corollary is a much more difficult question. Note that in dimension 1, this follows from the work of McMullen [Mc] and that a stronger statement using only the modulus of the multipliers has been recently given in [JX] using non-archimedean methods.

Corollary 2.2. *Let $d \geq 2$ and $k \geq 1$. If $(m_n)_{n\geq 1}$, $(x_n)_{n\geq 1}$ and $(\Lambda_n)_{n\geq 1}$ are as above with $X = \mathscr{M}_d^k$ then there exist $N \geq 1$ and a Zariski closed proper subset Γ of $\mathscr{M}_d^k$ such that Λ_n is finite-to-one on $\mathscr{M}_d^k \setminus \Gamma$ for all $n \geq N$.*

Proof. We simply apply Proposition 2.1 with f_{t_1} equal to an isolated Lattès map. By Berteloot and Dupont [BD1], Lattès maps are the only minimum of the Lyapunov function L so if f_{t_1} is an isolated Lattès map, there is no stable family in $\mathscr{M}_d^k$ containing it.

To construct such map, take $g\colon \mathbb{P}^1 \to \mathbb{P}^1$ an isolated Lattès map of degree d (which always exists by Milnor [Mi]), and if f_{t_1} is the k-symmetric power of g, we claim f_{t_1} is an isolated Lattès map in $\mathscr{M}_d^k$. Indeed, by [Yo, ALA], any Lattès map $f\colon \mathbb{P}^k \to \mathbb{P}^k$ comes from an isogeny $I : A \to A$ of an abelian variety A which is isogenous to E^k, where E is an elliptic curve. If f_{t_1} were not isolated, then it would be approximated by Lattès maps that in turn could be written as $f_{g'}$ but then, g' would approximate g which contradicts the fact that $g : \mathbb{P}^1 \to \mathbb{P}^1$ is an isolated Lattès map. □

2.3. **Families of regular polynomial endomorphisms of the affine plane.** In order to apply Proposition 2.1 on the moduli space $\mathscr{P}_d^2$, we prove the following rigidity result.

Theorem 2.3. *Let $d \geq 2$. Let g be a rational map of $\mathbb{P}^1$ of degree d which is not a flexible Lattès map. Moreover, assume that*

i) g possesses at least 3 postcritical repelling periodic points,
ii) for one of this postcritical repelling periodic point y_1 we have that

$$\{a \in \mathbb{P}^1\,;\ \textit{there exists } n \geq 1,\ g^n(a) = y_1 \ \textit{ and } \ a \ \textit{ is not in the critical set of } \ g^n\}$$

is dense in the Julia set J_g of g.

Let $(\mathbb{P}^2 \times Z, f, \mathcal{O}_{\mathbb{P}^2}(1))$ be a stable family of regular polynomial endomorphisms of $\mathbb{C}^2$ of degree d, parametrized by an irreducible algebraic curve Z. If there exists $\lambda_0 \in Z$ such that f_{λ_0} is equal to the lift of g to $\mathbb{C}^2$ then the family $(\mathbb{P}^2 \times Z, f, \mathcal{O}_{\mathbb{P}^2}(1))$ is isotrivial.

To the best of our knowledge, this is the first rigidity result in higher dimension which is not a direct consequence of one-dimensional results. However, the assumptions are probably far to be sharp. In particular, the point *ii)* above could be probably easily removed.

Proof. We will use several results of Bedford-Jonsson on regular polynomial endomorphisms of $\mathbb{C}^2$ obtained in [BJ].

Let λ be in Z. If $\mathrm{Crit}_{f_\lambda}$ denotes the critical set of f_λ in $\mathbb{P}^2$, we set $C_\lambda := \overline{\mathrm{Crit}_{f_\lambda} \setminus L_\infty}$. The critical measure of f_λ is $\mu_{c,\lambda} := T_\lambda \wedge [C_\lambda]$, where T_λ is the Green current of f_λ. In $\mathbb{C}^2$, $T_\lambda = \widehat{T}|_{\mathbb{P}^2 \times \{\lambda\}}$ is equal to the dd^c of the Green function G_λ of f_λ, which is non-negative on $\mathbb{C}^2$ and positive exactly outside the set K_λ of points of $\mathbb{C}^2$ with bounded orbit. Bedford-Jonsson proved in particular that the sum $L(\lambda)$ of the Lyapunov exponents of the equilibrium measure $\mu_\lambda := T_\lambda \wedge T_\lambda$ verifies

$$L(\lambda) = \log d + \ell(\lambda) + \int G_\lambda \mu_{c,\lambda},$$

where $\ell(\lambda)$ is the Lyapunov exponent associated to $f_{\lambda|L_\infty}$.

As the family is stable, $\lambda \mapsto L(\lambda)$ is a harmonic function on Z. Since it is positive, it has to be constant. Actually, both maps $\lambda \mapsto \ell(\lambda)$ and $\tilde{\ell}\colon \lambda \mapsto \int G_\lambda \mu_{c,\lambda}$ are also constant. To see this, first remark that the critical set $\mathrm{Crit}_f \subset \mathbb{P}^2 \times Z$ of the family can be decomposed as $\mathrm{Crit}(f) = (L_\infty \times Z) \cup C_f$. Moreover, by [BB1, P], the bifurcation current $dd^c L$ is equal to $\pi_{Z*}(\widehat{T}_f^2 \wedge [\mathrm{Crit}(f)])$, see § 2.1. It follows easily that $dd^c \ell = \pi_{Z*}(\widehat{T}_f^2 \wedge [Z \times L_\infty])$ and $dd^c \tilde{\ell} = \pi_{Z*}(\widehat{T}_f^2 \wedge [C_f])$. Hence, both ℓ and $\tilde{\ell}$ are subharmonic. They are also positive and their sum is harmonic on Z. Thus, they are constant.

This has two consequences. First, at the parameter λ_0 the map f_{λ_0} is the lift of g to $\mathbb{P}^2$ so $L(\lambda_0) = \log d + \ell(\lambda_0)$, i.e. $\tilde{\ell}(\lambda_0) := \int G_{\lambda_0} \mu_{c,\lambda_0} = 0$. Hence, $\tilde{\ell}(\lambda) = 0$ for all $\lambda \in Z$. In other words, the critical measure $\mu_{c,\lambda}$ is supported in K_λ. On the other hand, $(f_{\lambda|L_\infty})_{\lambda \in Z}$ is an algebraic stable family of rational maps on $\mathbb{P}^1$ so by [Mc] it has to be isotrivial since $f_{\lambda_0|L_\infty}$ is not a flexible Lattès map. Up to a finite cover of Z, we can then do a family of affine conjugations in order to have for each $\lambda \in Z$

- $f_{\lambda|L_\infty} = g$,
- $0 \in \mathbb{C}^2$ is a fixed point of f_λ, which is the continuation of the center of the pencil of curves preserved by f_{λ_0}.

Observe that there is still one degree of freedom, corresponding to homothety of center 0, which will be used latter.

We denote by X' the set of preperiodic critical points of g and by Y' its set of postcritical periodic points. The subset $Y \subset Y'$ corresponds to repelling postcritical periodic points and $X \subset X'$ to points which are eventually sent in Y. We also choose two integers $N \geq 1$ and $m \geq 1$ such that $g^N(X') \subset Y'$ and $g^m(y) = y$ for each $y \in Y'$. Observe that by i) the

set Y has at least 3 points. From that, the proof has four main steps. Notice that in what follows, we identify $\mathbb{P}^1$ with L_∞.

(1) For each λ, each irreducible component of the critical set of f_λ which contains a point of X has to be preperiodic.
(2) The periodic irreducible components of the postcritical set of f_λ passing through points of Y are lines containing 0. In other words, there exists a set of at least 3 lines $\mathcal{L} = \{L_y\,;\, y \in Y\}$ where each L_y is f_λ-periodic for all $\lambda \in Z$.
(3) The pencil of lines $\mathcal{P}$ passing through 0 has to be preserved by each f_λ.
(4) Up to homothety, there is a unique regular polynomial of $\mathbb{C}^2$ preserving $\mathcal{P}$ acting as g on L_∞.

Let us prove these four claims. Observe that the delicate one is **(3)** and that our proof is strongly inspired by [Mc] where the difficulties coming from *unlabelled* holomorphic motion are highlighted. In our very special situation, we use the lamination coming from [BJ, Theorem 8.8] to overcome possible monodromy problems.

Proof of (1). Let x be in X i.e a critical point of g whose image by g^N is a repelling m-periodic point y. Let λ be in Z and let C be an irreducible component of C_λ passing through x. The point y must be of saddle type for f_λ, repelling in the direction of L_∞ and super-attracting in the transverse direction. In particular, it admits a local stable manifold $W^s_{y,loc}$. On the other hand, $\mu_{c,\lambda} = T_\lambda \wedge [C_\lambda]$ vanishes near x so $(f^n_{\lambda|C})_{n\geq 0}$ is a normal family near x. The saddle nature of y gives that the only possible limit value of $(f^{N+nm)}_{\lambda|C})_{n\geq 0}$ near x is the constant function equal to y, i.e. $(f^{N+nm)}_{\lambda|C})_{n\geq 0}$ converges to this constant function on a neighborhood V of x in C. This implies that $f^N_\lambda(V) \cap W^s_{y,loc}$ is a neighborhood of y in $W^s_{y,loc}$ and thus, $f^N_\lambda(C)$ is m-periodic.

Proof of (2). Let $\mathcal{L}' = \{L_y\,;\, y \in Y'\}$ be the periodic postcritical lines for f_{λ_0} in the pencil $\mathcal{P}$. Observe that $f^m_{\lambda_0|L_y}$ is conjugated to z^{d^m} with a Julia set S_y which is uniformly hyperbolic.

Let λ be close enough to λ_0. Let $y \in Y$ and let D be an irreducible component of the postcritical set of f_λ containing y. As we have already seen, D is m-periodic and D is locally equal to the stable manifold of y near this point. In particular, D intersects transversely L_∞ at y. Moreover, by [DS2, Lemma 6.2], $f^m_{\lambda|D}$ has a topological degree equal to d^m and thus, using a normalization, it has a unique measure of maximal entropy $m \log d$.

On the other hand, since λ is close to λ_0, D is close to a union $\cup_{y' \in I} L_{y'}$ with $I \subset Y'$. For each $y' \in I$, the holomorphic motion of $S_{y'}$ gives a hyperbolic set of entropy $m \log d$ for $f^m_{\lambda|D}$ and thus, by uniqueness of the measure, we have that I is reduced to a point and thus $I = \{y\}$. In particular, the intersection points of D with L_∞ are postcritical periodic points close to y. As this set of points is discrete in L_∞ and independent of λ, $D \cap L_\infty$ is reduced to $\{y\}$ for λ close enough to λ_0. Furthermore, we have seen that this intersection is transverse. Thus, D has degree 1.

Finally, as λ is close to λ_0, 0 is attracting and D intersects its basin of attraction. By invariance of D, 0 is in D.

This proves the result in an euclidean neighborhood of λ_0. Since it is a closed property in the Zariski topology, it holds for all $\lambda \in Z$.

Proof of (3). Let denote by L_1 the line of $\mathcal{P}$ containing $y_1 \in Y$, the postcritical periodic point of g given by *ii)*. As we have seen, L_1 is m-periodic for all f_λ, $\lambda \in Z$. Since the Julia

set of $f^m_{\lambda_0|L_1}$ is contained in the small Julia set of f_{λ_0}, by [BBD, Theorem 1.1] the family $(f^m_{\lambda|L_1})_{\lambda\in Z}$ is stable. Indeed, in dynamics in dimension 2 J-stability is equivalent to the fact that J-repelling periodic points move holomorphically and remain repelling. In particular, all repelling periodic points of $f^m_{\lambda|L_1}$ move holomorphically and remain repelling. By [Mc], it has to be isotrivial. Hence, each $f^m_{\lambda|L_1}$ is holomorphically conjugated to $f^m_{\lambda_0|L_1}$, i.e. to $z\mapsto z^{d^m}$. In particular, up to a finite cover of Z and using a family of homotheties of $\mathbb{C}^2$, we can assume that $f^m_{\lambda|L_1}$ is independent of λ.

Let fix $\lambda\in Z$ for a moment and let denote by A_λ the basin of L_∞, i.e. $A_\lambda := \mathbb{P}^2\setminus K_\lambda$. As $\int G_\lambda\mu_{c,\lambda}=0$, by [BJ, Theorem 8.8], there exists a f_λ-invariant lamination by holomorphic discs $\{W_{a,\lambda}\,|\,a\in J_g\}$ in A_λ parametrized by the Julia set J_g of g, such that $W_{a,\lambda}\setminus\{a\}$ is either contained in the critical set of f_λ or disjoint from it. Moreover, $W_{a,\lambda}$ is contained in the stable manifold of a for a generic $a\in J_g$. Here, genericity is with respect to the equilibrium measure of $f_{\lambda|L_\infty}=g$. For the point y_1 defined above, $W_{y_1,\lambda}$ corresponds to the basin of attraction of y_1 for $f^m_{\lambda|L_1}$. As we have seen, this set is independent of λ and we denote it by W_{y_1}.

Let $n\geq 1$ and let $a_n\in g^{-n}(y_1)$ such that a_n is not in the critical set of g^n and $a_n\notin Y$. Observe that the set $W_{a_n,\lambda}$ satisfies $f^n_\lambda(W_{a_n,\lambda})=W_{y_1}$ (and thus is contained in the algebraic set $f^{-n}_\lambda(L_1)$) and that $W_{a_n,\lambda}\setminus\{a_n\}$ is disjoint from the critical set of f^n_λ. Let $w\in W_{y_1}\setminus\{y_1\}$. The set $P:=\{z\in W_{a_n,\lambda_0}\,|\,f^n_{\lambda_0}(z)=w\}$ has exactly d^n points. Moreover, if γ is a loop in Z and $z\in P$ then the fact that $W_{a_n,\lambda}\setminus\{a_n\}$ is disjoint from the critical set of f^n_λ ensures that we can follow z along γ as a point in $f^{-n}_\lambda(w)\cap W_{a_n,\lambda}$. This gives an action of $\pi_1(Z,\lambda_0)$ on P by permutations. But, $\pi_1(Z,\lambda_0)$ is generated by a finite set of elements $\{\gamma_1,\dots,\gamma_s\}$ thus the subgroup $H_n:=\langle\gamma_1^{d^n!},\dots,\gamma_s^{d^n!}\rangle$ has finite index and acts trivially on P. On the finite branch cover Z_n associated to H_n, the points in P can be followed holomorphically, i.e. there exists a family $(\phi_z)_{z\in P}$ of holomorphic maps from Z_n to $\mathbb{P}^2$ such that $\phi_z(\lambda_0)=z$ and $f^n_\lambda(\phi_z(\lambda))=w$ for all $z\in P$ and $\lambda\in Z_n$.

From this, there are two key observations. First, $W_{a_n,\lambda}$ is disjoint from L_y for $y\in Y$ since $a_n\notin Y$. Hence, if $\pi\colon\mathbb{P}^2\setminus\{0\}\to L_\infty$ denotes the linear projection, $\psi_z:=\pi\circ\phi_z$ defines maps from Z_n to $\mathbb{P}^1\setminus Y$. The other important observation is that the branch cover Z_n is independent of the choice of $w\in W_{y_1}\setminus\{y_1\}$. Thus, for each $z\in W_{a_n,\lambda_0}$ we can associated $\phi_z\colon Z_n\to\mathbb{P}^2\setminus\{L_y\,;\,y\in Y\}$ and $\psi_z:=\pi\circ\phi_z\colon Z_n\to\mathbb{P}^1\setminus Y$. As the set of non-constant holomorphic maps from Z_n to $\mathbb{P}^1\setminus\{y_1,\dots,z_k\}$ is finite and that $z\mapsto\psi_z(\lambda)$ is continuous for each $\lambda\in Z_n$, the maps ψ_z are either all constant or all equal. In both case, the fact that $\phi_z(\lambda)$ converges to a_n when $z\to a_n$ implies that each ϕ_z is constant equals to a_n. In other words, $W_{a_n,\lambda}$ is contained in $L_{a_n}:=\overline{\pi^{-1}(a_n)}$. Since the set of all possible a_n for all possible $n\geq 1$ is dense in J_g, each map f_λ satisfies $\pi\circ f_\lambda=g\circ\pi$ on $\pi^{-1}(J_g)$. This set is not pluripolar in $\mathbb{P}^2\setminus\{0\}$ so $\pi\circ f_\lambda=g\circ\pi$ on $\mathbb{P}^2\setminus\{0\}$, i.e. f_λ must preserve the pencil of line $\mathcal{P}$ defined by π.

Proof of (4). Let $\lambda\in Z$. Since f_λ preserves $\mathcal{P}$, it is of the form

$$f_\lambda[x:y:z]=[P(x,y):Q(x,y):R_\lambda(x,y,z)],$$

where $g[x,y]=[P(x,y):Q(x,y)]$. But f_λ is also a regular polynomial endomorphism of $\mathbb{C}^2$ so $R_\lambda(x,y,z)=c_\lambda z^d$. □

Remark 2.4. A similar result can probably be proved in higher dimension, i.e. for an algebraic family $(\mathbb{P}^k\times Z,f,\mathcal{O}_{\mathbb{P}^k}(1))$ of regular polynomial endomorphisms of $\mathbb{C}^k$ which are stable in the sense of [BBD]. There are two main difficulties. The first one is the lack

of rigidity results for the family $(H_\infty \times Z, f, \mathcal{O}_{\mathbb{P}^k}(1)|_{H_\infty})$ where H_∞ is the hyperplane at infinity. This can be easily overcome by choosing $f_{\lambda_0|H_\infty}$ to be a non-flexible Lattès map in higher dimension. We still have that the sum of the Lyapunov exponents associated to $f_{\lambda|H_\infty}$ is independent of λ. Hence, using the characterization [BD1] of Lattès maps, we obtain that the induced family on the hyperplane at infinity is isotrivial. The second difficulty is probably more serious and is related to the hyperbolicity of the triply punctured Riemann sphere. Results about the hyperbolicity of the complement of several hypersurfaces in projective space are known and a generalization of **(2)** above should be sufficient to obtain such hyperbolicity. However, this could take some work.

In particular, Corollary 2.2 also holds on $\mathscr{P}_d^2$.

Corollary 2.5. *Let $d \geq 2$. Let $(m_n)_{n\geq 1}$, $(x_n)_{n\geq 1}$ and $(\Lambda_n)_{n\geq 1}$ be as in Section 2.2 whith $X = \mathscr{P}_d^2$. There exist $N \geq 1$ and a Zariski closed proper subset Γ of $\mathscr{P}_d^2$ such that Λ_n is finite-to-one on $\mathscr{P}_d^2 \setminus \Gamma$ for all $n \geq N$.*

Proof. The proof is the same than Corollary 2.2 except that for the map f_{t_1} we take the lift f_{λ_0} from Theorem 2.3. Observe that in order to find a rational map g as in this theorem, it is sufficient to take a polynomial map with a postcritical repelling point of period 5. □

3. Blenders and the bifurcation measures

Our goal here is to prove that an open set Ω of $\mathscr{M}_d^k$ or $\mathscr{P}_d^2$ which satisfies a large set of assumptions (see § 3.2) has to be contained in the support of the bifurcations measure. More precisely, if it is not the case then by Theorem 3.4 below, Ω contains in a dense way positive dimensional subvarieties where the eigenvalues of all the periodic points on the small Julia set are constant. This contradicts Corollary 2.2 or Corollary 2.5. Observe that unlike the rest of the article, the families we consider here can be transcendental and are not necessarily closed. Except for Theorem 3.4 which only holds for $k = 2$ in the polynomial case, all the proofs in this section are the same for $\mathscr{M}_d^k$ and for $\mathscr{P}_d^k$. Hence, we focus our attention on the case of $\mathscr{M}_d^k$. The only difference for $\mathscr{P}_d^k$ is that the dimensions below, N_d^k and $\mathcal{N}_d^k$, have to be replaced by the dimension of Poly_d^k (i.e. $k\binom{k+d}{d}$) and the dimension of $\mathscr{P}_d^k$ (i.e. $k\left(\binom{k+d}{d} - (k+1)\right)$) respectively.

Actually, in this section and in Section 4, we will not work directly on $\mathscr{M}_d^k$ or $\mathscr{P}_d^k$. One major reason is that in Section 4 we will consider degenerations outside the space of endomorphisms of $\mathbb{P}^k$. In order to obtain a non-empty open subset in the support of the bifurcation measure on $\mathscr{M}_d^k$, we will provide a non-empty open subset $\Omega \subset \mathrm{End}_d^k$ in the support of the current $T_{f,\mathrm{Crit}}^{\mathcal{N}_d^k}$.

At the beginning of this section, after some basic notations, we give in § 3.2 a long list of assumptions that will be required in what follows. Then, in § 3.3 we state the main results of the whole section, Theorem 3.3 and Theorem 3.4, and explain how, combined to Corollary 2.2 or Corollary 2.5 and Theorem 4.1, they imply Theorem C. In § 3.4, we give the strategy of the proof of these two theorems and the structure of the remaining part of the section.

3.1. Notations.

If $a > 0$ then $\mathbb{D}_a$ is the disc of center 0 and radius a in $\mathbb{C}$ and we set $\mathbb{D} := \mathbb{D}_1$.

If $A \subset \mathbb{C}$ and $B \subset \mathbb{C}^{k-1}$ are two connected open subsets then $\Gamma \subset A \times B$ is a *vertical graph* if there exists a holomorphic function $g\colon B \to A$ with $\overline{g(B)} \Subset A$ and such that

$\Gamma = \{(g(y), y)\ ;\ y \in B\}$. One way to measure the verticality of a graph is to consider cone fields. As we will only work on $\mathbb{C}^k$ where the tangent bundle is trivial, for $\rho > 0$ we say that a vertical graph Γ as above is *tangent* to the cone field

$$C_\rho := \left\{(u_1, \dots, u_k) \in \mathbb{C}^k\ ;\ \rho|u_1| \leq \max_{2\leq i\leq k} |u_k|\right\}$$

if the tangent bundle $T\Gamma$ is contained in $\Gamma \times C_\rho$. If we write $\Gamma = \{(g(y), y)\ ;\ y \in B\}$ then this is equivalent to the fact that the partial derivatives of g are uniformly bounded by $1/\rho$. Observe that the more ρ is large, the more Γ is "vertical". The case $\rho = +\infty$ corresponds to vertical linear hyperspace. And we will say that a map f *contracts* the cone field C_ρ if there exists $\rho' > \rho$ such that the image of C_ρ by the differential of f at each point is contained in $C_{\rho'}$.

3.2. **Assumptions.** Let Ω be a non-empty open subset of End_d^k or of Poly_d^k such that each $f \in \Omega$ has the following properties. Observe that most of the objects below (all except J_k) are assumed to depend holomorphically on $f \in \Omega$ and the notations display this dependency. For example, $p\colon f \mapsto p(f)$ is the holomorphic motion of the saddle point given in Assumption 3 and Λ corresponds to the holomorphic motion of the hyperbolic set $\Lambda(f)$ from Assumption 2, i.e. $x \in \Lambda$ is a function $f \mapsto x(f)$ given by this holomorphic motion. Another observation is that the most important case is $k = 2$. When $k \geq 3$, the dynamics on the last $k-2$ coordinates is not very important. However, the point iii) in Assumption 10 prevents us from taking product maps.

(1) There exist two disjoint holomorphic discs $U_+, U_- \subset \mathbb{C}$ and two constants $R > 2$, and $\rho > 10$ such that f contracts the cone field C_ρ on $\mathcal{U} := \mathcal{U}_+ \cup \mathcal{U}_-$ where
$$\mathcal{U}_\pm := \mathbb{D}_R \times U_\pm \times \mathbb{D}^{k-2}.$$
(2) There exist two disjoint holomorphic discs $V_+, V_-, \subset \mathbb{C}$ such that, if we set
$$\mathcal{V}_+ := \mathbb{D}_R \times V_+ \times \mathbb{D}^{k-2},\quad \mathcal{V}_- := \mathbb{D}_R \times V_- \times \mathbb{D}^{k-2} \ \text{ and } \ \mathcal{V} := \mathcal{V}_+ \cup \mathcal{V}_-$$
then
- $\overline{V}_\pm \subset U_\pm$,
- f^2 contracts the cone field C_ρ on $\mathcal{V}$,
- f^2 is injective on $\mathcal{V}_\pm$ and $\overline{\mathcal{V}} \subset \overline{\mathcal{U}} \subset f^2(\mathcal{V}_\pm)$.

Moreover,
$$\Lambda(f) := \cap_{n\geq 0} f^{-2n}(\mathcal{V})$$
is a repelling hyperbolic set for f^2, contained in $J_k(f)$.

(3) f has a non-critical saddle fixed point $p(f) \in \mathbb{D} \times U_- \times \mathbb{D}^{k-2}$ with one stable direction and $k-1$ unstable directions. We ask that its local unstable manifold naturally extends to a vertical graph (denoted $W^u_{p(f),loc}$) in $\mathbb{D} \times V_- \times \mathbb{D}^{k-2}$ tangent to C_ρ. In what follows, $W^s_{p(f),loc}$ stands for a holomorphic disc in the stable manifold where f is conjugated to a contraction. Finally, we assume that f is C^1-linearizable near $p(f)$, with a linearization map depending continuously on f in the C^1-topology.

(4) Each vertical graph in $\mathbb{D}\times V_+ \times \mathbb{D}^{k-2}$ (resp. $\mathbb{D}\times V_- \times \mathbb{D}^{k-2}$) tangent to C_ρ intersects $\Lambda(f)$.

(5) The intersections between $W^u_{p,loc}$ and Λ are not persistent in Ω (i.e. if $x(f) \in W^u_{p(f),loc} \cap \Lambda(f)$ then there exists $g \in \Omega$ close to f such that the continuation $x(g)$ of $x(f)$ in $\Lambda(g)$ is not in $W^u_{p(g),loc}$).

(6) There exists a repelling 2-periodic point $r(f) \in \mathbb{D} \times V_- \times \mathbb{D}^{k-2}$ such that the eigenvalues of $D_{r(f)}f^2$ are all simple with no resonance. In particular, f^2 is linearizable near $r(f)$ and we assume that the domain of linearization contains $\overline{\mathcal{U}_-}$.
(7) There exists $n_0 \geq 1$ such that $f^{n_0}(\mathrm{Crit}(f))$ has a transverse intersection with $W^s_{p(f),loc} \setminus \{p(f)\}$.
(8) There exist $K \in \mathbb{N}$ and $\tilde{q}(f) \in W^u_{p(f),loc}$ which is not a critical point for f^K and such that $q(f) := f^K(\tilde{q}(f)) \neq p(f)$ is a transverse homoclinic intersection in $W^s_{p(f),loc}$.
(9) The exceptional set of f is disjoint from its small Julia set.

It remains a last assumption which is more technical. It will be used to initiate the induction on the dimension. Let $\chi_{p(f)}$ (resp. $\chi_{r(f)}$) be the eigenvalue of $D_{p(f)}f$ (resp. $D_{r(f)}f^2$) with the smallest modulus.

(10) For every non-empty open subset $\Omega' \subset \Omega$, there exists $f \in \Omega'$, $m \in \mathbb{N}$ and $x(f) \in \Lambda(f)$ such that
  i) $f^m(x(f)) = r(f)$,
  ii) $x(f) \in W^u_{p(f),loc}$,
  iii) $D_{x(f)}f^m(T_{x(f)}W^u_{p(f),loc})$ is a "generic" hyperplane for $D_{r(f)}f^2$, i.e. does not contain any eigenvector of $D_{r(f)}f^2$,
  iv) the subgroup $\langle \chi_{p(f)}, \chi_{r(f)} \rangle$ of $\mathbb{C}^*$ generated by $\chi_{p(f)}$ and $\chi_{r(f)}$ is dense,

From a non-technical point of view, the main ingredients to obtain that $\Omega \subset \mathrm{supp}(T^{\mathcal{N}^k_d}_{f,\mathrm{Crit}})$ are Assumptions 2 to 4. They should be sufficient for the proof. Assumption 4 says that $\Lambda(f)$ satisfies a blender property and by Assumption 3, there exists a connection between this blender $\Lambda(f)$ and the saddle point $p(f)$. If the critical set has a transverse intersection with the stable manifold of $p(f)$, this gives rise by the inclination lemma to infinite intersections between the postcritical set and $\Lambda(f)$. Very likely, all these intersections should provide as many as possible independent bifurcations. Most of the other assumptions are there in order to ensure several transversality properties which eventually give the existence of these independent bifurcations. In particular, a transverse intersection between $W^s_{p(f),loc}$ and the postcritical set is given by Assumption 7. Observe moreover that this assumption also implies easily that Ω contains no PCF maps (see the end of the proof of Theorem C or [Le, Corollary 2.5] for a more precise result).

In the example we construct, all these assumptions are easy to check, except Assumption 10. This last assumption is the key technical point to prove that the support of the bifurcation measure has non-empty interior. Establishing it on Ω takes a large part of Section 4 where we need to consider degenerations outside End^k_d.

In order to give more explanations on this assumption, the point iv) will be used to ensure that the postcritical set of f can approximate any leaf of a foliation by hypersurfaces $\mathcal{F}_f$, defined in a neighborhood of $r(f)$ as the vertical fibration associated to the linearization map. The point iii) implies in particular that the strong unstable hyperplane $T_{r(f)}W^{uu}_{r(f)}$ together with $(D_{x(f)}f^{m+i}(T_{x(f)}W^u_{p(f),loc}))_{1\leq i\leq k-1}$ from a basis of hyperplanes. Each of them is actually the tangent space of a dynamical foliation, $\mathcal{F}_f$ and $(\mathcal{G}^i_f)_{1\leq i\leq k-1}$ respectively, which thus define local coordinates near $r(f)$. A key point will be that, under suitable conditions labeled as $(\star)$ in what follows, these coordinates provided local conjugacies which turn out to extend in a neighborhood of the small Julia set. The results of Section 2 ensure then that the conjugacies are generically global.

Finally, notice that it would be easier to work with a fixed point in Assumption 6. However, we were not able to obtain the open set Ω when $d = 2$ with this additional constraint.

3.3. **Statements.** Here, we assume that Ω satisfies all the assumptions of § 3.2. The purpose of Assumption 10 is to construct families with the following properties.

Definition 3.1. *A subvariety $M \subset \Omega$ satisfies the condition* (†) *if*

(1) *M is connected,*
(2) *there exists $x \in \Lambda$ and $m \in \mathbb{N}$ such that for all $f \in M$, $x(f) \in W^u_{p(f),loc}$, $f^m(x(f)) = r(f)$ and $D_{x(f)}f^m(T_{x(f)}W^u_{p(f),loc})$ is a generic hyperplane for $D_{r(f)}f^2$,*
(3) *all the intersections points in $W^u_{p(f)} \cap \Lambda(f)$ can be followed holomorphically,*
(4) *there exists $f_0 \in M$ such that the subgroup $\langle \chi_r(f_0), \chi_p(f_0)\rangle$ is dense in $\mathbb{C}^*$.*

We also consider a stronger condition.

Definition 3.2. *A subvariety $M \subset \Omega$ satisfies the condition* ($\star$) *if it is simply connected, verifies* (†) *and it is a stable family in the sense of Berteloot-Bianchi-Dupont.*

The main purpose of this whole section is to show that these conditions combined to the assumptions on Ω bring us to these two results.

Theorem 3.3. *If $M \subset \Omega$ satisfies* (†) *then the functions $f \mapsto \chi_{p(f)}$ and $f \mapsto \chi_{r(f)}$ are constant on M. In particular, any connected analytic subset $M' \subset M$ also satisfies* (†).

Theorem 3.4. *Let $M \subset \Omega$ be an analytic subset which satisfies* ($\star$)*. Let $f_0 \in M$. Then each n-periodic point $x(f_0)$ of $J_k(f_0)$ can be followed as a n-periodic point $x(f)$ in $J_k(f)$ and the map $f \mapsto \det D_{x(f)}f^n$ is constant on M.*

From this, if we anticipate the existence of the open set Ω, established in Theorem 4.1, we can conclude the proof of Theorem C.

Proof of Theorem C. We only consider the case of $\mathscr{M}^k_d$. As we already said, the proof for $\mathscr{P}^2_d$ is exactly the same except that $\mathcal{N}^k_d$ has to be replaced by $2\binom{d+2}{d} - 6$. Observe that we cannot conclude the proof on $\mathscr{P}^k_d$ when $k \geq 3$ since Corollary 2.5 only holds on $\mathscr{P}^2_d$.

Let $k \geq 2$ and $d \geq 2$. Let Ω be the open subset of End^k_d given by Theorem 4.1. Our goal is to show that $\Omega \subset \mathrm{supp}(T^{\mathcal{N}^k_d}_{f,\mathrm{Crit}})$. To that end, we consider a non-empty connected open subset $\Omega' \subset \Omega$ and we will prove that $\Omega' \cap \mathrm{supp}(T^{\mathcal{N}^k_d}_{f,\mathrm{Crit}})$ is not empty. First, fix an arbitrary element $f' \in \Omega'$. If we apply Corollary 2.2 to the sequence $(x_n)_{n\geq 1}$ of all repelling periodic points of f' in $J_k(f')$ then there exists $N \geq 1$ such that the corresponding multiplier map Λ_N is generically finite on a branch cover of $\mathscr{M}^k_d$. As the periodic points $(x_n)_{1\leq n\leq N}$ are repelling and in $J_k(f')$, they can be followed holomorphically as repelling points in $J_k(f)$ in a small neighborhood of f' in End^k_d. Since Λ_N is generically finite, the fibers of the corresponding map on a small open subset $\Omega'' \subset \Omega'$ close to f' have codimension $\mathcal{N}^k_d$. Hence, by Theorem 3.4, any analytic subset of Ω'' satisfying ($\star$) must have codimension at least $\mathcal{N}^k_d$.

Now, by Assumption 10 there exists $f_0 \in \Omega''$, $m \in \mathbb{N}$ and $y_1(f_0) \in \Lambda(f_0)$ such that

i) $f_0^m(y_1(f_0)) = r(f_0)$,
ii) $y_1(f_0) \in W^u_{p(f_0),loc}$,
iii) $D_{y_1(f_0)}f_0^m(T_{y_1(f_0)}W^u_{p(f_0),loc})$ is a generic hyperplane for $D_{r(f_0)}f_0^2$,

iv) the subgroup $\langle \chi_{p(f_0)}, \chi_{r(f_0)} \rangle$ of $\mathbb{C}^*$ is dense.

In particular, f_0 belongs to

$$A_1 := \{f \in \Omega'' \ ; \ y_1(f) \in W^u_{p(f),loc}\},$$

which is a hypersurface by Assumption 5. A priori, A_1 might be non-irreducible which will give rise to mild technical difficulties in what follows. However, the parameters f in A_1 such that $\langle \chi_{p(f)}, \chi_{r(f)} \rangle$ is dense in $\mathbb{C}^*$ is itself dense in any irreducible component of A_1 containing f_0. Actually, if $P(f)$ and $R(f)$ denote logarithms of $\chi_{p(f)}$ and $\chi_{r(f)}$ and if we write $R(f) = t(f)P(f) + \theta(f)2i\pi$ with $t(f), \theta(f) \in \mathbb{R}$ then this condition on the subgroup is equivalent to the fact that 1, $t(f)$ and $\theta(f)$ are independent over $\mathbb{Q}$. This holds outside a countable union of real analytic subsets of A_1. Hence, we can take a smooth point f_1 of A_1 such that this condition is satisfies and such that f_1 is close enough to f_0 to ensure that the point iii) above also holds for f_1. Let X_1 be the irreducible component of A_1 containing f_1 and let $\Omega_1 \subset \Omega''$ be a small open neighborhood of f_1 such that $A_1 \cap \Omega_1 = X_1 \cap \Omega_1$ and iii) is satisfied on Ω_1.

The set $W^u_{p(f)} \cap \Lambda(f_1)$ is infinite and we use it to define a sequence $(f_i)_{1 \leq i \leq N}$ in Ω'', a decreasing sequence $(\Omega_i)_{1 \leq i \leq N}$ of open subsets of Ω'', and a decreasing sequence $(X_i)_{1 \leq i \leq N}$ of smooth irreducible analytic sets such that $\mathrm{codim}(X_i) = i$ and $f_i \in X_i$. The construction goes as follow. Assume that $(f_i)_{1 \leq i \leq i_0}$, $(\Omega_i)_{1 \leq i \leq i_0}$ and $(X_i)_{1 \leq i \leq i_0}$ are defined. If all the intersections points in $W^u_{p(f_i)} \cap \Lambda(f_i)$ can be followed holomorphically on X_{i_0} then we set $N := i_0$ and the construction ends. Otherwise, there exist $n_{i_0+1} \geq 0$ and $y_{i_0+1} \in \Lambda$ such that $y_{i_0+1}(f_{i_0}) \in f_{i_0}^{n_{i_0+1}}(W^u_{p(f_{i_0}),loc})$ and this relation is not persistent on X_{i_0}. Here, since $f_{i_0}^{n_{i_0+1}}(W^u_{p(f_{i_0}),loc})$ is not a closed analytic set, we mean that there exists a small neighborhood $\tilde{\Omega}_{i_0+1} \subset \Omega_{i_0}$ of f_{i_0} such that

$$A_{i_0+1} := \{f \in X_{i_0} \cap \tilde{\Omega}_{i_0+1} \ ; \ y_{i_0+1}(f) \in f^{n_{i_0+1}}(W^u_{p(f),loc})\},$$

is a closed hypersurface in $X_{i_0} \cap \tilde{\Omega}_{i_0+1}$. As above, the set A_{i_0+1} might be non-irreducible but we can choose a smooth point f_{i_0+1} on it such that $\langle \chi_{p(f_{i_0+1})}, \chi_{r(f_{i_0+1})} \rangle$ is dense in $\mathbb{C}^*$ and such that the condition iii) holds for f_{i_0+1}. We then choose X_{i_0+1} to be the irreducible component of A_{i_0+1} which contains f_{i_0+1} and we take $\Omega_{i_0+1} \subset \tilde{\Omega}_{i_0+1}$ to be a small enough neighborhood of f_{i_0+1} to have iii) on it and also $A_{i_0+1} \cap \Omega_{i_0+1} = X_{i_0+1} \cap \Omega_{i_0+1}$. Observe that we always have $N \leq \mathcal{N}^k_d$ and that X_N satisfies (†).

Another observation is that X_N corresponds to N independent intersections between $W^u_{p(f)}$ and $\Lambda(f)$ and, since the J_k-repelling periodic points are dense in Λ and since by Assumption 7 some parts of the postcritical set approximate $W^u_{p(f)}$, a small perturbation of X_N gives rise to an analytic set which corresponds to N-properly J_k-prerepelling parameters. As this point is important, we give more details. Let $(y_i)_{1 \leq i \leq N}$ and $(n_i)_{1 \leq i \leq N}$ be as above with the convention that $n_1 = 0$. Moreover, during the construction, we used open subsets $(\Omega_i)_{i \leq i \leq N}$ from which we define $\tilde{\Omega} := \cap_{i=i}^N \Omega_i$. We now consider the sets

$$W_N := \{(f, z_1, \ldots, z_N) \in \tilde{\Omega} \times (\mathbb{P}^k)^N \ ; \ z_i \in f^{n_i}(W^u_{p(f),loc}) \text{ for } 1 \leq i \leq N\}$$

and

$$Y_N := \{(f, z_1, \ldots, z_N) \in \tilde{\Omega} \times (\mathbb{P}^k)^N \ ; \ z_i = y_i(f) \text{ for } 1 \leq i \leq N\}.$$

What we have proved so far is that the projection of $W_N \cap Y_N$ on $\tilde{\Omega}$, which is equal to $X_N \cap \tilde{\Omega}$, has codimension N. Since the projection of Y_N on $\tilde{\Omega}$ is a biholomorphism, we

have that $W_N \cap Y_N$ has pure dimension $N_d^k - N$, where $N_d^k := \dim(\mathrm{End}_d^k)$. On the other hand, $\dim(Y_N) = N_d^k$ and $\dim(W_N) = N_d^k + (k-1)N$ so we have

$$\dim(W_N \cap Y_N) = \dim(W_N) + \dim(Y_N) - \dim(\tilde{\Omega} \times (\mathbb{P}^k)^N),$$

which gives the properness of the intersection. As the repelling periodic points are dense in Λ, the set Y_N is approximated by sets $Y'_{N,n}$ defined in the same way replacing each y_i by repelling periodic points $y'_{i,n}$ converging to y_i. Moreover, the inclination lemma and Assumption 7 also give that W_N is approximated by sets $W'_{N,n}$ defined as W_N but using a local branch of some iterate of the critical set instead of $W^u_{p(f),loc}$. The persistence of proper intersections (see e.g. [C, §12.3]) gives that $W'_{N,n} \cap Y'_{N,n}$ is proper when n is large enough i.e. $W'_{N,n} \cap Y'_{N,n}$ corresponds to N-properly J_k-prerepelling points in $\tilde{\Omega} \times (\mathbb{P}^k)^N$.

Now, we continue the construction and we define by induction $(f_i)_{N+1\le i\le N'}$, $(\Omega_i)_{N+1\le i\le N'}$ and $(X_i)_{N+1\le i\le N'}$ in the following way. Assume the construction done for $N \le i \le i_0$. If the family defined by X_{i_0} is stable then we set $N' := i_0$. Otherwise, there exists a non-persistent Misiurewicz relation on X_{i_0} and we define A_{i_0+1} to be the analytic hypersurface of X_{i_0} where this relation is persistent. Then, we choose a smooth point f_{i_0+1} on A_{i_0+1} and a small neighborhood $\Omega_{i_0+1} \subset \Omega_{i_0}$ such that $X_{i_0+1} := \Omega_{i_0+1} \cap A_{i_0+1}$ is smooth, connected and simply connected.

As above, at the end we have

- $\mathrm{codim}(X_{N'}) = N' \le \mathcal{N}_d^k$,
- all the Misiurewicz relations in $X_{N'}$ are persistent, i.e. this family is stable,
- by Theorem 3.3 $X_{N'}$ satisfies (†) and thus ($\star$).

The construction of Ω'' and Theorem 3.4 then ensure that $N' \ge \mathcal{N}_d^k$ and thus $N' = \mathcal{N}_d^k$. On the other hand, exactly as above, the points of $X_{N'}$ are approximated by N'-properly J_k-prerepelling parameters in $\mathrm{End}_d^k \times (\mathbb{P}^k)^N$. By Proposition 1.11, $X_{N'}$ is contained in the support of the current $T^{\mathcal{N}_d^k}_{f,\mathrm{Crit}}$. Moreover, the bifurcation measure $\mu_{f,\mathrm{Crit}}$ of the moduli space $\mathscr{M}_d^k$ satisfies $\Pi^*(\mu_{f,\mathrm{Crit}}) = T^{\mathcal{N}_d^k}_{f\mathrm{Crit}}$, where $\Pi : \mathrm{End}_d^k \to \mathscr{M}_d^k$ is the natural projection, see [BB1]. This implies $\hat{\Omega} := \Pi(\Omega'') \subset \mathrm{supp}(\mu_{f,\mathrm{Crit}})$ and $\hat{\Omega}$ is open since Π is an open map.

Finally, notice that Assumption 7 easily gives that the open set Ω obtained by Theorem 4.1 doesn't possess PCF maps. More precisely, let $f \in \Omega$. The inclination lemma applied to the piece of $f^{n_0}(\mathrm{Crit}_f)$ transverse to $W^s_{p(f),loc}$ given by Assumption 7 implies that the postcritical set contains infinitely many disjoint (local) hypersurfaces converging to $W^u_{p(f),loc}$. Hence, the postcritical set cannot be algebraic. □

Remark 3.5. *For the absence of PCF maps in Ω we could have use a result of Le* [Le, Corollary 2.5] *saying that a PCF map of $\mathbb{P}^k$ cannot have non-critical saddle periodic point.*

3.4. **Sketch of the proofs of Theorems 3.3 and 3.4.** Let $M \subset \Omega$ be a subvariety satisfying ($\star$). The assumptions of § 3.2 are used in the following way.

(1) As the family is stable, by [Bia, Theorem C] there exists an equilibrium lamination $\mathcal{L}$ for the family $(f)_{f\in M}$ (see Definition 3.19).
(2) Points **(2)** and **(3)** in Definition 3.1, which come from Assumption 10, ensure that $r(f) \in W^u_{p(f)}$ persistently in the family.
(3) Since the postcritical set intersects transversely $W^s_{p(f)}$ (Assumption 7), the inclination lemma and the assumption $\overline{\langle \chi_{p(f_0)}, \chi_{r(f_0)} \rangle} = \mathbb{C}^*$ implies that the postcritical

set of f_0 can approximate any leaf of a foliation by hypersurfaces $\mathcal{F}_{f_0}$, defined in a neighborhood of $r(f_0)$ as the vertical fibration associated to the linearization map.

(4) The previous point, the stability of $(f)_{f\in M}$ and the blender property from Assumption 4 imply, in a first time, a persistent relation $\chi_{r(f)} = \zeta\chi_{p(f)}^{-\omega}$ on M which actually gives, combined to Assumption 8, that both these functions are constant.

(5) The genericity part of **(2)** in Definition 3.1 allows us to construct $k-1$ other local foliations, $\mathcal{G}_f^1, \dots, \mathcal{G}_f^{k-1}$ whose leaves are also approximated by the postcritical set and such that $(\mathcal{F}_f, \mathcal{G}_f^1, \dots, \mathcal{G}_f^{k-1})$ provides local coordinates near $r(f)$.

(6) The fact that the equilibrium lamination $\mathcal{L}$ is acritical implies that if $\gamma \in \mathcal{L}$ then the coordinates of $\gamma(f)$ with respect to $(\mathcal{F}_f, \mathcal{G}_f^1, \dots, \mathcal{G}_f^{k-1})$ are independent of f.

(7) Since $\{\gamma(f) \mid \gamma \in \mathcal{L}\}$ is not contained in a proper analytic set, these local coordinates respecting $\mathcal{L}$ give a local conjugacy near $r(f)$.

(8) This local conjugacy extends to a neighborhood of the small Julia set, forcing the multipliers to be constant in the family.

Now, in § 3.5 we set notations and basic results for the family $(f)_{f\in\Omega}$. § 3.6 and § 3.7 are devoted to obtain the points 3 and 4 which actually imply Theorem 3.3. The conjugacy, which corresponds to points 5 to 8, is constructed in § 3.8.

3.5. **Semi-local dynamics.** First, we fix an arbitrary $f_0 \in \Omega$. As Theorem 3.4 is essentially a local result, when necessary we will replace Ω by a smaller connected open neighborhood of f_0 in Ω.

Since the fibration of $\mathbb{C}^k$ by vertical hypersurfaces will play an important role in what follows, we denote by $\pi\colon \mathbb{C}^k \to \mathbb{C}$ the first projection.

Let $r(f)$ be the repelling 2-periodic point given by Assumption 6. We denote by $\chi_{r(f)}$ the eigenvalue of $D_{r(f)}f^2$ with the smallest modulus. Since the eigenvalues of $D_{r(f)}f^2$ have no resonance, by [BR] there exist a holomorphic family of holomorphic maps $(\phi_f)_{f\in\Omega}$ from $\mathbb{C}^k$ to $\mathbb{P}^k$ and a holomorphic family $(\tilde{L}_f)_{f\in\Omega}$ of linear self-maps of $\mathbb{C}^{k-1}$ such that $\phi_f(0) = r(f)$ and

$$\phi_f^{-1} \circ f^2 \circ \phi_f(x,y) = (\chi_{r(f)}x, \tilde{L}_f(y)) =: L_f(x,y)$$

for every $(x,y) \in \mathbb{C}\times\mathbb{C}^{k-1}$ near 0. In particular, the vertical linear fibration of $\mathbb{C}^k$ defined by π is sent on the strong unstable fibration of $r(f)$ and $\phi_{f|\pi^{-1}(0)}$ gives a parametrization of the strongly unstable manifold of $r(f)$. Moreover, Assumption 6 implies that there exists a neighborhood A of 0 in $\mathbb{C}^k$ such that ϕ_{f_0} is injective on A with $\overline{\mathcal{U}_-} \subset \phi_{f_0}(A)$. The cone condition in Assumption 2 ensures that there is an open set $\tilde{A} \subset \mathbb{C}^{k-1}$ such that $\{0\}\times\tilde{A} \subset A$ and $\phi_{f_0}(\{0\}\times\tilde{A})$ is a vertical graph which goes through $\mathbb{D}\times V_-\times\mathbb{D}^{k-2}$. As these properties are stable under small perturbations, there exists $\nu > 0$ such that, possibly by reducing Ω and slightly $A, \tilde{A}$, for each $f \in \Omega$

- ϕ_f is injective on A with $\overline{\mathcal{U}_-} \subset \phi_f(A)$,
- for all $c \in \mathbb{D}_\nu$, $\{c\}\times\tilde{A} \subset A$ and $\phi_f(\{c\}\times\tilde{A})$ is a vertical graph passing through $\mathbb{D}\times V_-\times\mathbb{D}^{k-2}$.

We denote by $\delta_f\colon \phi_f(A) \to A$ the associated inverse map. Observe that the second point above combined to Assumption 4 implies that each $\phi_f(\{c\}\times\tilde{A})$ intersects $\Lambda(f)$. In what follows, it will be convenient to normalize the family $(\phi_f)_{f\in\Omega}$ in the following way. Let consider a family $(u_f)_{f\in\Omega}$ of self-maps of $\mathbb{C}\times\mathbb{C}^{k-1}$ of the form $u_f(x,y) = (a_f x, y)$ where $a_f \in \mathbb{C}^*$ depends holomorphically on f and is chosen such that

- $|a_f| < \nu$ in order to have $u_f(\mathbb{D}\times\tilde{A}) \subset \mathbb{D}_\nu\times\tilde{A}$,

- there exists $r' \in \Lambda$ close enough to r such that for all f in Ω, $\pi \circ u_f^{-1} \circ \delta_f(r'(f)) \equiv 1$.

Hence, if we replace ϕ_f by $\phi_f \circ u_f$ then ν can be supposed to be 1 and we have $\pi \circ \delta_f(r'(f)) \equiv 1$. This normalization will only appear in Corollary 3.18 which is, however, a key ingredient in § 3.8.

We then set $B := \mathbb{D} \times \tilde{A}$ and $D_f := \phi_f(B)$. The later possesses a natural foliation $\mathcal{F}_f$ where $\mathcal{F}_f(c) = \phi_f(\{c\} \times \tilde{A})$ for $c \in \mathbb{D}$. As we have already seen, each leaf is a vertical graph which intersects $\Lambda(f)$. In particular, $\mathcal{F}_f(0)$ corresponds to the local strongly unstable manifold $W^{uu}_{r(f),loc}$ of $r(f)$. We also denote by $W^{cu}_{r(f),loc} = \phi_f(\mathbb{D} \times \{0\})$, which corresponds to the local weak unstable manifold of $r(f)$.

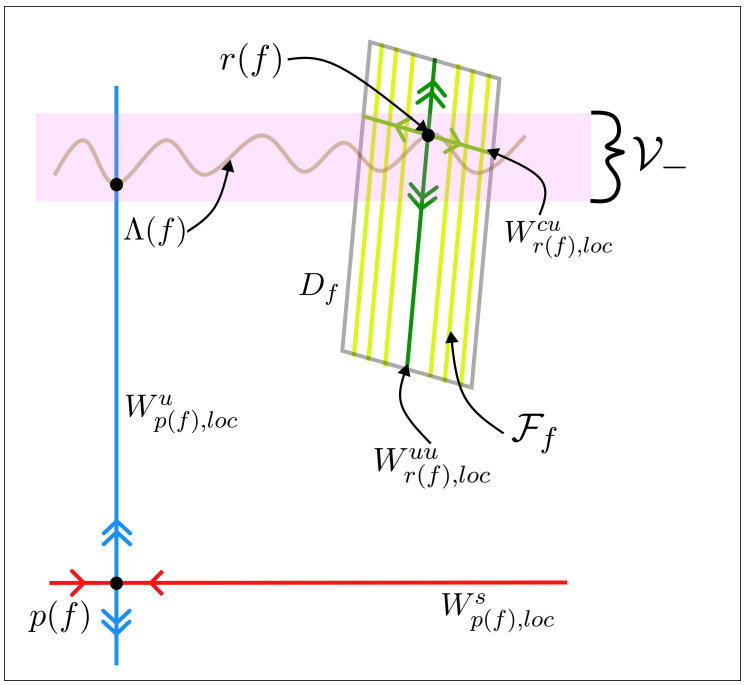

FIGURE 1. Summary of the notations. The whole picture is contained in $\mathcal{U}_-$. In the example we obtain in Section 4, the hyperbolic set $\Lambda(f)$ is a Cantor set but it intersects any sufficiently vertical graph in $\mathcal{V}_-$.

Finally, we also set some notations about the dynamics near the saddle fixed point $p(f)$ given by Assumption 3. Let denote by $\chi_{p(f)}$ the eigenvalue of $D_{p(f)}f$ with the smallest modulus. Using holomorphic conjugacies separately on the stable/unstable directions, we first choose holomorphic local coordinates $v_f \colon V_f \to \mathbb{D}^k$ on a neighborhood V_f of $p(f)$ such that

- $v_f^{-1}(\mathbb{D} \times \{0\})$ is contained in the stable manifold of $p(f)$ and $v_f \circ f \circ v_f^{-1}(x, 0) = (\chi_{p(f)} x, 0)$,
- $v_f^{-1}(\{0\} \times \mathbb{D}^{k-1})$ is contained in the unstable manifold of $p(f)$ and, when it is defined, $v_f \circ f \circ v_f^{-1}(0, y) = (0, A_f(y))$ where A_f is an expanding diagonal matrix of size $k-1$.

In what follows, the local stable manifold of $p(f)$ will be defined as $W^s_{p(f),loc} := v_f^{-1}(\mathbb{D} \times \{0\})$. For the local unstable manifold $W^u_{p(f),loc}$ of $p(f)$, we take the vertical graph in $\mathbb{D} \times V_- \times \mathbb{D}^{k-2}$ given by Assumption 3.

Moreover, this assumption also implies that there exists a C^1-family $(\theta_f)_{f\in M}$ of local C^1-diffeomorphisms such that $\theta_f \circ f \circ \theta_f^{-1}$ is the linear map $K_f(x,y) = (\chi_{p(f)}x, A_f(y))$. Observe that we can assume the domain of definition of θ_f contains V_f and that $D_0(\theta_f \circ v_f^{-1}) = \mathrm{id}$.

Consequences of the inclination lemma. We will extensively use the inclination lemma on families of hypersurfaces transverse to $W^s_{p(f),loc}$ parametrized by a subset M of Ω. We will gradually strengthen the assumptions on M until reaching (†) in § 3.7 and ($\star$) in § 3.8. For now, we just assume that M is a connected analytic subset of Ω.

Definition 3.6. *We say that $\Gamma = (\Gamma(f))_{f\in M}$ is a family of polydiscs intersecting transversely $W^s_{p(f),loc}$ at $b(f)$ if*

- *each $\Gamma(f)$ is biholomorphic to $\mathbb{D}^{k-1}$ and $f \mapsto \Gamma(f)$ is holomorphic,*
- *each $\Gamma(f)$ intersects $W^s_{p(f),loc}$ in a unique point and this intersection is transverse,*
- *the image by v_f of this intersection point with $\Gamma(f)$ is $(b(f), 0) \in \mathbb{D} \times \{0\}$.*

From now on, we also assume there exists $x \in \Lambda$ and $m \in \mathbb{N}$ such that for all $f \in M$, $x(f) \in W^u_{p(f),loc}$, $f^m(x(f)) = r(f)$ and $D_{x(f)}f^m(T_{x(f)}W^u_{p(f),loc})$ is a generic hyperplane for $D_{r(f)}f^2$. By increasing m if necessary, we can assume that f^m sends biholomorphically a neighborhood of $x(f)$ in $W^u_{p(f),loc}$ to a vertical graph $W_m(f)$ in D_f, C^1-close to $W^{uu}_{r(f),loc}$ and thus tangent to C_ρ.

Let Γ be a family of polydiscs intersecting transversely $W^s_{p(f),loc}$. By the inclination lemma, there exists $k_0 \geq 0$ such that for all $f \in M$ and all $k \geq k_0$, $f^k(\Gamma(f))$ contains a subset which is C^1-close to $W^u_{p(f),loc}$. In particular, $f^{k+m}(\Gamma(f))$ contains a subset $\Gamma_k(f)$ which is a vertical graph in D_f, tangent to C_ρ and C^1-close to $W_m(f)$. From this, using again the inclination lemma but near $r(f)$, we can construct families of vertical graphs which turns out to be key objects to prove that (†) implies a persistent resonance between $\chi_p(f)$ and $\chi_{r(f)}$.

Definition 3.7. *Let Γ and k_0 be as above. Let $f \in M$, $l_0 \geq 0$ and $k \geq k_0$. Let $c_k(f)$ denote the point of intersection between $\Gamma_k(f)$ and $W^{cu}_{r(f),loc}$. If for all $0 \leq l \leq l_0$, $|\pi \circ \delta_f(f^{2l}(c_k(f)))| < 1/2$ then we define $\Gamma_{k,l}(f)$ by induction, setting*

- *$\Gamma_{k,l}(f)$ is the vertical graph in $f^2(\Gamma_{k,l-1}) \cap D_f$ which contains $f^{2l}(c_k(f))$.*

In this situation, we say that $\Gamma_{k,l}(f)$ is well-defined for all $l \leq l_0$.

Remark 3.8. (1) Observe that the injectivity in Assumption 2 implies that there is no ambiguity in the definition of $\Gamma_{k,l}(f)$.
(2) A priori, it could happen that $\Gamma_{k,l}(f)$ is well defined for some $f = f_1$ and not for $f = f_2$, even if $|\pi \circ \delta_{f_1}(f_1^{2l}(c_k(f_1)))|$ is much smaller than $1/2$. However, we will see in Lemma 3.12 that under condition (†), this doesn't happen and thus $(\Gamma_{k,l}(f))_{f\in M}$ define holomorphic families of vertical graphs.

Since $D_{x(f)}f^m(T_{x(f)}W^u_{p(f),loc})$ is a generic hyperplane for $D_{r(f)}f^2$, in particular $W^{cu}_{r(f),loc}$ is transverse to $W_m(f)$. Hence, by the inclination lemma there exist an integer $a > m$ and a holomorphic injective map $h_f \colon \mathbb{D} \to V_f$ (where V_f is the neighborhood of $p(f)$ defined above) such that

- $\Delta_f := h_f(\mathbb{D})$ is transverse to $W^u_{p(f),loc}$,
- Δ_f is a graph above $W^s_{p(f),loc}$, more precisely the projection on the first coordinate of $v_f \circ h_f$ is the identity,
- $f^a_{|\Delta_f}$ is injective and $f^a(\Delta_f)$ is a neighborhood of $r(f)$ in $W^{cu}_{r(f),loc}$.

We define $H_f \colon \mathbb{D} \to \mathbb{D}$ by $H_f = \pi \circ \delta_f \circ f^a \circ h_f$. Observe that $H_f(0) = 0$ and is injective. Hence, there exists $\tilde{\alpha}(f) \neq 0$, which depends holomorphically on f, such that

$$H_f(s) = \tilde{\alpha}(f)s + o(s),$$

where $o(s)$ is uniform in f.

Remark 3.9. Observe that a similar construction can be done where $W^{cu}_{r(f),loc}$ is replaced by a holomorphic disc Σ_f transverse to $f^n(W^u_{p(f),loc})$ as long as the point in $W^u_{p(f),loc}$ sent to $\Sigma_f \cap f^n(W^u_{p(f),loc})$ is not critical for f^n. We will use such construction in Proposition 3.17 for the homoclinic intersection given by Assumption 8, i.e. Σ_f will be an open subset of $W^s_{p(f),loc}$.

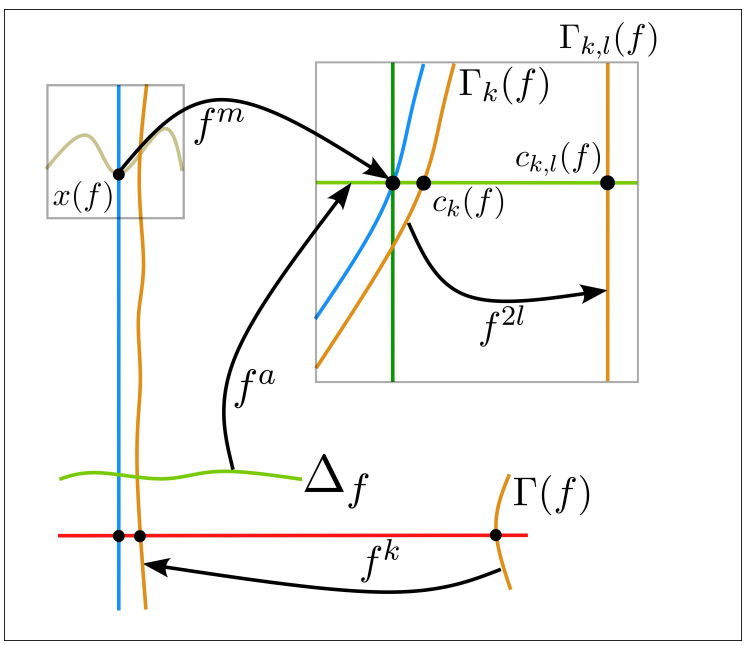

FIGURE 2. Definition of $\Gamma_{k,l}(f)$ where $x(f) \in W^u_{p(f),loc}$ is a preimage of $r(f)$. The integers m and a are constant but k and l can be large. The next two lemmas show that $c_k(f)$ and $c_{k,l}(f)$ are essentially equal to $\chi_p(f)^k$ and $\chi_p(f)^k\chi_r(f)^l$ respectively in the coordinates on $W^{cu}_{r(f),loc}$ given by ϕ_f.

The two following lemmas can be seen as consequences of the inclination lemma or linearization results. Their purpose is to show that the vertical graphs $\Gamma_{k,l}(f)$ are close to leafs $\mathcal{F}_f(c_{k,l}(f))$ of the strong unstable foliation of $r(f)$, where $c_{k,l}(f)$ is essentially equal to $\chi_p(f)^k\chi_r(f)^l$. The first lemma focus on $\Gamma_k(f)$. It should be possible to prove it using distorsion estimates. We instead use C^1-linearization and this part of Assumption 3 only appears here.

Lemma 3.10. *There exists a holomorphic function $\beta\colon M \to \mathbb{C}^*$ with the following property. Let $(\Gamma(f))_{f\in M}$ be a family of polydiscs intersecting transversely $W^s_{p(f),loc}$ at $b(f)$. For each $n \geq 0$ large enough there exists a holomorphic function $s_n\colon M \to \mathbb{C}$ such that for each $f \in M$*

- $\Delta_f \cap f^n(\Gamma(f)) = h_f(s_n(f))$,
- $s_n(f) = \beta(f)b(f)\chi^n_{p(f)} + o(\chi^n_{p(f)})$.

In other words, for $k \geq 0$ large enough, $\Gamma_k(f) \cap W^{cu}_{r(f),loc} \in \mathcal{F}_f(H_f(s_{k+m-a}(f)))$ with

$$H_f(s_{k+m-a}(f)) = \alpha(f)b(f)\chi^k_{p(f)} + u_k(f),$$

with $\alpha(f) := \chi^{m-a}_{p(f)}\tilde{\alpha}(f)\beta(f)$ and such that $u_k(f)/\chi^k_{p(f)}$ converges to 0, locally uniformly on M.

Proof. As θ_f is well defined on V_f, we can define a C^1-germ of $(\mathbb{C}, 0)$ by

$$F_f(s) := \pi \circ v_f \circ \theta_f^{-1} \circ \tilde{\pi}_0 \circ \theta_f \circ h_f(s),$$

where $\pi\colon \mathbb{C}^k \to \mathbb{C}$ is the first projection as above and $\tilde{\pi}_0\colon \mathbb{C}^k \to \mathbb{C}^k$ is defined by $\tilde{\pi}_0(x, y) = (x, 0)$. The key point in the proof is that the differential of F_f at 0 is $\mathbb{C}$-linear. The maps h_f and $\pi\circ v_f$ are holomorphic so we focus on $\theta_f^{-1}\circ\tilde{\pi}_0\circ\theta_f$ which corresponds to the projection in the unstable direction given by the linearization θ_f. This maps is a priori not holomorphic but there exists a sequence $(U_n)_{n\geq 0}$ of open neighborhoods of $W^u_{p(f),loc} \cap V_f$ such that

$$\pi_n := f^n \circ v_f^{-1} \circ \tilde{\pi}_0 \circ v_f \circ f^{-n}$$

is defined on U_n. We claim that, for $z \in W^u_{p(f),loc} \cap V_f$, we have

$$\pi_n(z) = 0 = \theta_f^{-1} \circ \tilde{\pi}_0 \circ \theta_f(z) \quad \text{and} \quad D_z\pi_n \xrightarrow[n\to\infty]{} D_z(\theta_f^{-1} \circ \tilde{\pi}_0 \circ \theta_f).$$

Indeed, the equality is obvious and for the convergence, if we set $\psi := \theta_f \circ v_f^{-1}$ and $F := v_f \circ f \circ v_f^{-1}$ then

$$\theta_f \circ \pi_n \circ \theta_f^{-1} = \psi \circ F^n \circ \tilde{\pi}_0 \circ F^{-n} \circ \psi^{-1} = K_f^n \circ \psi \circ \tilde{\pi}_0 \circ \psi^{-1} \circ K_f^{-n}.$$

It follows that, if y satisfies $\theta_f(z) = (0, y)$ then, using that $D_0\psi = \mathrm{id}$ and writing $D_{(0,A_f^{-n}(y))}\psi^{-1} = \mathrm{id} + E_n$, we have

$$D_{(0,y)}(\theta_f \circ \pi_n \circ \theta_f^{-1}) = K_f^n \circ \tilde{\pi}_0 \circ (\mathrm{id} + E_n) \circ K_f^{-n} = K_f^n \circ \tilde{\pi}_0 \circ K_f^{-n} + K_f^n \circ \tilde{\pi}_0 \circ E_n \circ K_f^{-n}.$$

K_f commutes with $\tilde{\pi}_0$ thus the first term is equal to $\tilde{\pi}_0$. The second one converges to 0 since $\|K_f^{-n}\| \simeq \chi^{-n}_{p(f)}$, $\|K_f^n \circ \tilde{\pi}_0\| \simeq \chi^n_{p(f)}$ and $\|E_n\|$ converges to 0.

This gives that $\theta_f^{-1}\circ\tilde{\pi}_0\circ\theta_f$ is $\mathbb{C}$-differentiable on $W^u_{p(f),loc}$ and so at $h_f(0)$. Hence, there exists $\gamma(f) \in \mathbb{C}$, which is non-zero since Δ_f and $W^s_{p(f),loc}$ are transverse to $W^u_{p(f),loc}$, such that $F_f(s) = \gamma(f)s + o(s)$.

On the other hand, let $(\Gamma(f))_{f\in M}$ be a family of polydiscs intersecting transversely $W^s_{p(f),loc}$ at $b(f)$. For each $n \in \mathbb{N}$ large enough, there exists $s_n(f) \in \mathbb{D}$ which depends holomorphically on $f \in M$ and such that $h_f(s_n(f)) \in f^n(\Gamma(f)) \cap \Delta_f$. The set $\tilde{\Gamma}(f) := \theta_f(\Gamma(f))$ is locally a vertical graph $\{(g_f(y), y)\}$ where $g_f\colon (\mathbb{C}^{k-1}, 0) \to \mathbb{C}$ is a C^1-germ. Its image $\tilde{\Gamma}^n(f)$ by K_f^n is given locally by $\{(g_f^n(y), y)\}$ where $g_f^n(y) = \chi^n_{p(f)} g_f(A_f^{-n}(y))$. Hence, since A_f^{-n} contracts at an exponential speed, we have $g_f^n(y) = \chi^n_{p(f)} g_f(0) + o(\chi^n_{p(f)})$ where the

error term is uniform in f. Moreover, there exists $y_n(f)$ such that $\theta_f^{-1}(g_f^n(y_n(f)), y_n(f)) = h_f(s_n(f))$. Therefore, the definitions of v_f and K_f give

$$\begin{aligned}
&F_f(s_n(f)) = \pi \circ v_f \circ \theta_f^{-1} \circ \tilde{\pi}_0(g_f^n(y_n(f)), y_n(f))) = \pi \circ v_f \circ \theta_f^{-1}(g_f^n(y_n(f)), 0) \\
&= \pi \circ v_f \circ \theta_f^{-1}(\chi_{p(f)}^n g_f(0), 0) + o(\chi_{p(f)}^n) = \pi \circ v_f \circ \theta_f^{-1}(K_f^n(\theta_f(v_f^{-1}(b(f), 0)))) + o(\chi_{p(f)}^n) \\
&= \pi \circ v_f \circ f^n(v_f^{-1}(b(f), 0)))) + o(\chi_{p(f)}^n) = \chi_{p(f)}^n b(f) + o(\chi_{p(f)}^n).
\end{aligned}$$

Hence, $s_n(f) = \beta(f)b(f)\chi_{p(f)}^n + o(\chi_{p(f)}^n)$ where $\beta(f) := \gamma(f)^{-1}$. To conclude, the sequence $s_n(f)/(b(f)\chi_{p(f)}^n)$ depends holomorphically on f and this sequence converge locally uniformly to $\beta(f)$ which is then also holomorphic. □

The next lemma can be seen as a consequence of the inclination lemma in presence of a dominated splitting. It can be easily proved using the linearization near $r(f)$ and the proof is left to the reader.

Lemma 3.11. *Let $(\Gamma(f))_{f\in M}$ be a family of polydiscs intersecting transversely $W^s_{p(f),loc}$ at $b(f)$. There exists a sequence $(\epsilon_n)_{n\geq 0}$ converging to 0 with the following property. If $f \in M$, $k \geq n$ and $l \geq n$ are such that $\Gamma_{k,l}(f)$ is well-defined then*

$$d(\Gamma_{k,l}(f), \mathcal{F}_f(c_{k,l}(f))) \leq \epsilon_n,$$

where $c_{k,l}(f) = \alpha(f)b(f)\chi_{p(f)}^k \chi_{r(f)}^l$.

3.6. **Strong relations between the multipliers.** From now on, we consider a subvariety $M \subset \Omega$ which satisfies **(1)**, **(2)** and **(3)** in Definition 3.1. Once again, we fix $f_0 \in M$.

Lemma 3.12. *Let $(k_n)_{n\geq 0}$ and $(l_n)_{n\geq 0}$ be two increasing sequences such that $(\chi_{p(f_0)}^{k_n}\chi_{r(f_0)}^{l_n})_{n\geq 0}$ is a sequence in $\mathbb{D}$ which converges. Then $\{f \mapsto \chi_{p(f)}^{k_n}\chi_{r(f)}^{l_n}\}_{n\geq 0}$ is a normal family in a neighborhood $M_0 \subset M$ of f_0.*

Proof. A preliminary observation is that since Λ moves with respect to a holomorphic motion, the family of functions on M, $\{f \to x(f)\}_{x\in\Lambda}$ is a normal family. In particular, there exists a neighborhood $M_0 \subset M$ of f_0 such that if $x(f_0)$ is in D_{f_0} with $|\pi\circ\delta_{f_0}(x(f_0))| < \frac{1}{20}$ then for all $f \in M_0$, $x(f) \in D_f$ with $|\pi \circ \delta_f(x(f))| < \frac{1}{10}$.

By Assumption 8, there exists a family of polydiscs Γ intersecting transversely $W^s_{p(f),loc}$ at some $b(f) \neq 0$ and such that $\Gamma(f) \subset W^u_{p(f)}$. By exchanging each $\Gamma(f)$ by an appropriate subset of $f^N(\Gamma(f))$, we can assume that, for all $f \in M_0$, $|\alpha(f)b(f)| < \frac{1}{20}$.

By Lemma 3.10, the first coordinate (with respect to δ_{f_0}) of $c_{k_n}(f_0) := \Gamma_{k_n}(f_0) \cap W^{cu}_{r(f_0),loc}$ is $H_{f_0}(s_{k_n+m-a}(f_0)) = \alpha(f_0)b(f_0)\chi_{p(f_0)}^{k_n} + u_{k_n}(f_0)$. Hence,

$$\pi \circ \delta_{f_0}(f_0^{2l}(c_{k_n}(f_0))) = \chi_{r(f_0)}^l \left(\alpha(f_0)b(f_0)\chi_{p(f_0)}^{k_n} + u_{k_n}(f_0)\right).$$

Since $(\chi_{p(f_0)}^{k_n}\chi_{r(f_0)}^{l_n})_{n\geq 0}$ converges to some $\chi(f_0) \in \mathbb{D}$ and since $u_{k_n}(f_0)/\chi_{p(f_0)}^{k_n}$ converges to 0, the sets $\Gamma_{k_n,l_n}(f_0)$ are well-defined for $n \geq n_0$ for some n_0 large enough.

By Assumption 4, for $n \geq n_0$ there exists a point $x_n(f_0) \in \Lambda(f_0)$ which belongs to $\Gamma_{k_n,l_n}(f_0)$. On the other hand, by Lemma 3.11, the sequence of analytic sets $(\Gamma_{k_n,l_n}(f_0))_{n\geq n_0}$ converges to $\mathcal{F}_{f_0}(\alpha(f_0)b(f_0)\chi(f_0))$. Since $|\alpha(f_0)b(f_0)| < \frac{1}{20}$, this implies that $|\pi\circ\delta_{f_0}(x_n(f_0))| < \frac{1}{20}$ for $n \geq n_1$ large enough and thus $|\pi \circ \delta_f(x_n(f))| < \frac{1}{10}$ for all $f \in M_0$.

As M satisfies **(3)** in Definition 3.1, the persistence of proper intersections (see e.g. [C, §12.3]) implies that the continuation $x_n(f)$ of $x_n(f_0)$ in $\Lambda(f)$ also lies on $\Gamma_{k_n,l_n}(f)$, which is thus well-defined. As observe above, all these functions $\{f \mapsto x_n(f)\}_{n\geq n_1}$ form a normal family. Hence, the same holds for the family

$$\left\{ f \mapsto \frac{\pi \circ \delta_f(x_n(f))}{b(f)\alpha(f)} \right\}_{n\geq n_1}.$$

The result follows since, by Lemma 3.10 and Lemma 3.11, these functions are, locally on M_0, arbitrarily close to $\{f \mapsto \chi_{p(f)}^{k_n}\chi_{r(f)}^{l_n}\}_{n\geq 0}$. □

Proposition 3.13. *There exists $\zeta \in \mathbb{S}^1$ and $\omega \in \mathbb{R}_+^*$ such that for all $f \in M$*

$$\chi_{r(f)} = \zeta\chi_{p(f)}^{-\omega}.$$

Proof. Let $(k_n)_{n\geq 0}$ and $(l_n)_{n\geq 0}$ be two sequences as in Lemma 3.12 that we choose in order to have

$$\chi(f_0) := \lim_{n\to\infty} \chi_{p(f_0)}^{k_n}\chi_{r(f_0)}^{l_n}$$

is non-zero. By analytic continuation, it is sufficient to prove the result in a neighborhood of f_0. Let M_0 be the neighborhood of f_0 obtained by Lemma 3.12 where the family $\{f \mapsto \chi_{p(f)}^{k_n}\chi_{r(f)}^{l_n}\}_{n\geq 0}$ is normal. Let $\chi\colon M_0 \to \mathbb{C}$ be a limit value and we can assume, up to take a subsequence, that for each $f \in M_0$,

$$\chi_{p(f)}^{k_n}\chi_{r(f)}^{l_n} \to \chi(f). \tag{2}$$

Let M_1 be a simply connected neighborhood of f_0 where χ doesn't vanish. Let $P(f)$ (resp. $R(f)$, resp. $Q(f)$) be a logarithm of $\chi_{p(f)}$ (resp. $\chi_{r(f)}$, resp. $\chi(f)$) on M_1. By (2), the real parts of these functions verify on M_1

$$\lim_{n\to\infty} k_n \mathrm{Re}P(f) + l_n \mathrm{Re}R(f) = \mathrm{Re}Q(f)$$

and thus

$$\lim_{n\to\infty} \frac{k_n}{l_n}\mathrm{Re}P(f) + \mathrm{Re}R(f) = 0.$$

Hence, if ω denotes a limit value of $(k_n/l_n)_{n\geq 0}$ then $\mathrm{Re}R(f) = -\omega\mathrm{Re}P(f)$. This implies that there exists $t \in \mathbb{R}$ such that $R = -\omega P + it$ and so $\chi_{r(f)} = \zeta\chi_{p(f)}^{-\omega}$ with $\zeta := e^{it}$. □

This gives precise information on the possible limit values for families of the form $(\Gamma_{k,l})$ obtained by Definition 3.7.

Lemma 3.14. *There exist $t_0 > 0$ and a neighborhood $M_0 \subset M$ of f_0 with the following property. Let $(k_n)_{n\geq 0}$ and $(l_n)_{n\geq 0}$ be two increasing sequences such that $(\chi_{p(f_0)}^{k_n}\chi_{r(f_0)}^{l_n})_{n\geq 0}$ converges to $\xi\chi_{p(f_0)}^{t}$ for some $t \in [t_0, +\infty[$ and $\xi \in \mathbb{S}^1$. Let Γ be a family of polydiscs cutting $W^s_{p(f),loc}$ transversely at $b(f)$. Let $(\Gamma_{k,l})$ the associated sequence of families of polydiscs obtained by Definition 3.7. Then, there exists $n_0 \in \mathbb{N}$ such that for $f \in M_0$, $(\Gamma_{k_n,l_n}(f))_{n\geq n_0}$ is well defined and converges to $\mathcal{F}(\alpha(f)b(f)\xi\chi_{p(f)}^{t})$, uniformly on M_0.*

Proof. Let $t_0 > 0$ be such that $|\alpha(f_0)\chi_{p(f_0)}^{t_0}| < \frac{1}{20}$ and let $M_0 \subset M$ be a relatively compact neighborhood of M such that $|\alpha(f)\chi_{p(f)}^{t_0}| < \frac{1}{10}$ for all $f \in M_0$. By Proposition 3.13, there exist $\omega \in \mathbb{R}$ and ζ in $\mathbb{S}^1$ such that for all $f \in M$, $\chi_{r(f)} = \zeta\chi_{p(f)}^{-\omega}$. Hence, $\lim_{n\to\infty} \chi_{p(f_0)}^{k_n}\chi_{r(f_0)}^{l_n} = \xi\chi_{p(f_0)}^{t}$ implies that $(k_n - \omega l_n)_{n\geq 0}$ and $(\zeta^{l_n})_{n\geq 0}$ converge to t and ξ

respectively. Therefore, for all $f \in M$, $(\chi_{p(f)}^{k_n}\chi_{r(f)}^{l_n})_{n\geq 0}$ converges to $\xi\chi_{p(f)}^t$. As we assumed that $t \geq t_0$, we thus have that $|\alpha(f)b(f)\chi_{p(f)}^{k_n}\chi_{r(f)}^{l_n}| < \frac{1}{5}$ for all $f \in M_0$ and $n \geq n_1$ where $n_1 \in \mathbb{N}$ is large enough.

By Lemma 3.10, for all $f \in M$, $\Gamma_{k_n}(f)$ passes through $W^{cu}_{r(f),loc} \cap \mathcal{F}(H_f(s_{k_n+m-a}(f)))$, where $H_f(s_{k_n+m-a}(f)) = \alpha(f)b(f)\chi_{p(f)}^{k_n} + u_{k_n}(f)$ with $u_{k_n}(f)/\chi_{p(f)}^{k_n}$ converging to 0, uniformly on $f \in M_0$. Thus, for $n_0 \geq n_1$ large enough we have $|u_{k_n}(f)\chi_{r(f)}^{l_n}| < \frac{1}{5}$ on M_0 for all $n \geq n_0$. Hence, $|\chi_{r(f)}^{l_n}H_f(s_{k_n+m-a}(f))| < \frac{1}{2}$ and $(\Gamma_{k_n,l_n}(f))_{n\geq n_0}$ are well-defined. On the other hand, the convergence above implies that

$$\lim_{n\to\infty} \chi_{r(f)}^{l_n}H_f(s_{k_n+m-a}(f)) = \alpha(f)b(f)\xi\chi_{p(f)}^t.$$

This combined to Lemma 3.11 give that $(\Gamma_{k_n,l_n}(f))_{n\geq n_0}$ converges to $\mathcal{F}(\alpha(f)b(f)\xi\chi_{p(f)}^t)$. □

3.7. **Stereotyped holomorphic motion and constant multipliers.** From now on, we assume that $M \subset \Omega$ satisfies condition (†) and we choose an element $f_0 \in M$ such that $\overline{\langle \chi_{p(f_0)}, \chi_{r(f_0)}\rangle} = \mathbb{C}^*$.

Remark 3.15. Observe that this last point above is equivalent to the fact that t, ω and 1 are independent over $\mathbb{Q}$ where $\chi_{r(f_0)} = e^{2i\pi t}\chi_{p(f_0)}^{-\omega}$. Hence, Proposition 3.13 implies that $\overline{\langle \chi_{p(f)}, \chi_{r(f)}\rangle} = \mathbb{C}^*$ for all $f \in M$. This actually gives the last point in Theorem 3.3.

We prove now that this additional assumption on M constrains the holomorphic motion of $\Lambda(f)$ to be very special.

Proposition 3.16. *Let $(\Gamma(f))_{f\in M}$ be a family of polydiscs cutting $W^s_{p(f),loc}$ transversely at $b(f)$ such that $\Gamma(f) \subset W^u_{p(f)}$. Pick $x(f_0) \in \Lambda(f_0) \cap D_{f_0}$ such that $\pi(\delta_{f_0}(x(f_0))) = \alpha(f_0)b(f_0)\xi\chi_{p(f_0)}^t$ for some $t \in \mathbb{R}$ and $\xi \in \mathbb{S}^1$. Then, for all $f \in M$, the holomorphic continuation $x(f)$ in $\Lambda(f)$ of $x(f_0)$ is in D_f and satisfies $\pi(\delta_f(x(f))) = b(f)\alpha(f)\xi\chi_{p(f)}^t$.*

Proof. Let $t_0 > 0$ and $M_0 \subset M$ be as in Lemma 3.14. Let $x(f_0) \in \Lambda(f_0) \cap D_{f_0}$. By exchanging $x(f_0)$ by a preimage, we can assume that $\pi(\delta_{f_0}(x(f_0))) = \alpha(f_0)b(f_0)\xi\chi_{p(f_0)}^t$ with $t > t_0$. Since $\overline{\langle \chi_{p(f_0)}, \chi_{r(f_0)}\rangle} = \mathbb{C}^*$, there exist $(k_n)_{n\geq 0}$ and $(l_n)_{n\geq 0}$ two increasing sequences such that $(\chi_{p(f_0)}^{k_n}\chi_{r(f_0)}^{l_n})_{n\geq 0}$ converges to $\xi\chi_{p(f_0)}^t$. Let Γ_{k_n,l_n} be the families of analytic sets associated to $\Gamma(f) \subset W^u_{p(f)}$. Lemma 3.14 implies that $(\Gamma_{k_n,l_n}(f))_{\geq 0}$ converges to $\mathcal{F}(\alpha(f)b(f)\xi\chi_{p(f)}^t)$, uniformly on M_0. Hence, if $x(f)$ intersects properly $\mathcal{F}_f(\alpha(f)b(f)\xi\chi_{p(f)}^t)$ then $x(f)$ intersects properly $\Gamma_{k_n,l_n}(f)$ for $n \geq 0$ large enough. This contradicts condition (†) and thus $x(f) \in \mathcal{F}_f(\alpha(f)b(f)\xi\chi_{p(f)}^t)$ for all $f \in M_0$, i.e. $\pi(\delta_f(x(f))) = b(f)\alpha(f)\xi\chi_{p(f)}^t$. By analytic continuation, this equality holds on the whole space M. □

Using several homoclinic intersections, we obtain the following strong restriction on χ_p which, combined to Proposition 3.13, implies Theorem 3.3

Proposition 3.17. *The function χ_p is constant on M.*

Proof. Assume by contradiction that χ_p is not constant. This implies the existence of a small arc $\gamma\colon [0,1] \to M$ such that $\chi_{p(\gamma(s))} = re^{isa}$ for all $s \in [0,1]$, where $r \in \mathbb{C}^*$ and $a > 0$ are two constants. In particular, for $n \in \mathbb{N}$ large, $s \mapsto \chi_{p(\gamma(s))}^n$ spines about n/a times

around 0 on a small circle. We will use this fast variation in the argument together with Proposition 3.16 in order to obtain a contradiction.

By Assumption 8, there exist $K \geq 1$ and $\tilde{q}(f) \in W^u_{p(f),loc}$ which is not a critical point for f^K and such that $q(f) := f^K(\tilde{q}(f)) \neq p(f)$ is a transverse homoclinic intersection in $W^s_{p(f),loc}$. We denote by $b_0(f)$ the point in $\mathbb{D}$ such that $v_f(q(f), 0) = b_0(f)$ and we have $b_0(f) \neq 0$. As the homoclinic intersection is transverse, there exists a family of polydisc $(\Gamma(f))_{f\in M}$ intersecting transversely $W^s_{p(f),loc}$ at $b_0(f)$ such that $\Gamma(f) \subset W^u_{p(f)}$. Moreover, as observe in Remark 3.9, since $\tilde{q}(f)$ is not critical for f^K, there exists a holomorphic injective map $g_f \colon \mathbb{D} \to V_f$ such that

- $\Delta'_f := g_f(\mathbb{D})$ is transverse to $W^u_{p(f),loc}$,
- Δ'_f is a graph above $W^s_{p(f),loc}$, i.e. the projection on the first coordinate of $v_f \circ g_f$ is the identity,
- $f^K_{|\Delta'_f}$ is injective and $f^K(\Delta'_f)$ is a neighborhood of $q(f)$ in $W^s_{p(f),loc}$.

We also define $G_f \colon \mathbb{D} \to \mathbb{D}$ by $G_f = \pi \circ v_f \circ f^K \circ g_f$ where $\pi \colon \mathbb{C}^k \to \mathbb{C}$ is the first projection. Observe that G_f is injective with $G_f(0) = b_0(f)$. Hence, there exists $\tilde{\beta}(f) \neq 0$, which depends holomorphically on f, such that

$$G_f(s) = b_0(f) + \tilde{\beta}(f)s + o(s),$$

where $o(s)$ is uniform in f. Exactly like in Lemma 3.10, for $n \geq K$ large enough, there exist holomorphic functions s'_n and δ_n such that

- $g_f(s'_n(f)) \in f^{n-K}(\Gamma(f))$,
- $s'_n(f) = b_0(f)\chi^{n-K}_{p(f)} + o(\chi^n_{p(f)})$,
- $G_f(s'_n(f)) = b_0(f)(1 + \beta(f)\chi^n_{p(f)} + \delta_n(f))$, where $\delta_n(f) = o(\chi^n_{p(f)})$ and $\beta(f) := \tilde{\beta}(f)\chi^{-K}_{p(f)}$.

We set $b_n(f) := G_f(s'_n(f))$ which corresponds to a transverse homoclinic intersection $f^n(\Gamma(f)) \cap W^s_{p(f),loc}$ very close to $b_0(f)$.

Now, if we apply Proposition 3.16 first to $b(f) = b_n(f)$ with $\xi = 1$, $t = 0$ and a second time to $b(f) = b_0(f)$, where ξ_n and t_n are chosen such that $\xi_n\chi^{t_n}_{p(f_0)} = 1 + \chi^n_{p(f_0)}\beta(f_0) + \delta_n(f_0)$, then we obtain for all $f \in M$

$$(b_0(f) + b_0(f)\chi^n_{p(f)}\beta(f) + b_0(f)\delta_n(f))\alpha(f) = b_0(f)\alpha(f)\xi_n\chi^{t_n}_{p(f)},$$

and thus

$$1 + \chi^n_{p(f)}\beta(f) + \delta_n(f) = \xi_n\chi^{t_n}_{p(f)}. \tag{3}$$

Observe that ξ_n converges to 1 and t_n converges to 0 since $1 + \chi^n_{p(f_0)}\beta(f_0) + \delta_n(f_0)$ goes to 1.

We choose an arc $\gamma \colon [0, 1] \to M$ as above, small enough to insure that the argument of $s \mapsto \beta(\gamma(s))$ is almost constant and such that $\chi_{p(\gamma(s))} = re^{isa}$ for all $s \in [0, 1]$, where $r \in \mathbb{C}^*$ and $a > 0$ are two constants. In particular, when n is large then $s \mapsto \chi^n_{p(\gamma(s))}\beta(\gamma(s))$ spines about n/a times around 0 on a small circle.

On the other hand, let $P \colon \Omega \to \mathbb{C}$ be a logarithm of χ_p on Ω and let $\theta_n \in \mathbb{R}$ converging to 0 such that $\xi_n = e^{i\theta_n}$. Then $\xi_n\chi^{t_n}_{p(f)} - 1 = (t_nP(f) + i\theta_n) + o(t_nP(f) + i\theta_n)$ whose argument is essentially equal to the one of $t_nP(f) + i\theta_n$ which is never purely imaginary since $|\chi_{p(f)}| < 1$ on Ω. Hence, the equality (3) cannot hold for $n \geq 1$ large enough. This gives the desired contradiction. □

The combination of Proposition 3.17 with Proposition 3.16 easily says that the first coordinate (with respect to ϕ_f) of the holomorphic motion of points in $\Lambda(f) \cap D_f$ is not only holomorphic in $f \in M$ but also in the starting point. Actually, in the coordinates given by ϕ_f, this dependency is linear and our choice of normalization of ϕ_f implies that it is constant.

Corollary 3.18. *If $x(f_0) \in \Lambda(f_0) \cap D_f$ then for all $f \in M$*

$$\pi(\delta_f(x(f))) = \pi(\delta_{f_0}(x(f_0))).$$

Proof. As in the proof of Proposition 3.17, let $q(f)$ be the homoclinic intersection and $b_0(f)$ the corresponding point in $\mathbb{D}$. By Proposition 3.17 and Proposition 3.16, if

$$\pi(\delta_{f_0}(x(f_0))) = \alpha(f_0)b_0(f_0)s,$$

for some $s \in \mathbb{C}$, then

$$\pi(\delta_f(x(f))) = \alpha(f)b_0(f)s.$$

In other words

$$\pi(\delta_f(x(f))) = \pi(\delta_{f_0}(x(f_0)))\frac{b_0(f)\alpha(f)}{b_0(f_0)\alpha(f_0)}.$$

On the other hand, in order to normalize ϕ_f we had chosen $r' \in \Lambda$ in § 3.5 close enough to r such that

$$\pi(\delta_f(r'(f))) = \pi(\delta_{f_0}(r'(f_0))).$$

Hence, $\frac{b_0(f)\alpha(f)}{b_0(f_0)\alpha(f_0)}$ is constantly equal to 1. □

3.8. **Construction of the conjugacy.** We will first construct local conjugacies between elements of M and then extend them in a neighborhood of the small Julia set J_k. This problematic is classical in one variable complex dynamics. See in particular [BE1] where Buff-Epstein obtained at the end a global conjugacies outside the exceptional sets. In our context we have much less information on the dynamics outside the small Julia set and, even if the counterpart of [BE1] probably holds in higher dimension, our final argument relies strongly on the fact that we work with a family.

Let $M \subset \Omega$ be a subvariety which satisfies ($\star$). The difference with condition (†) is that $(f)_{f\in M}$ is supposed to be simply connected and stable in the sense of Berteloot-Bianchi-Dupont [BBD]. Observe that in [BBD, Theorem 1.1], the parameter space has to be an open subset of End_d^k. However, this restriction has been overcome by Bianchi in the broader setting of polynomial-like maps with large topological degree [Bia]. The key notion in [BBD, Bia] for what follows is the *equilibrium lamination*. To introduce it, we first consider the set

$$\mathcal{J} := \left\{\gamma\colon M \to \mathbb{P}^k \,\middle|\, \gamma \text{ is holomorphic and } \gamma(f) \in J_k(f) \text{ for every } f \in M\right\}.$$

The family $(f)_{f\in M}$ induces naturally a self-map F of $\mathcal{J}$ by setting $F(\gamma)(f) := f(\gamma(f))$.

Definition 3.19. *An* equilibrium lamination *is a relatively compact subset $\mathcal{L}$ of $\mathcal{J}$ such that*

(1) *$\gamma(f) \neq \gamma'(f)$ for all $f \in M$ if $\gamma, \gamma' \in \mathcal{L}$ with $\gamma \neq \gamma'$,*
(2) *for every $f \in M$, the equilibrium measure of f gives full mass to $\{\gamma(f) \mid \gamma \in \mathcal{L}\}$,*
(3) *for every $f \in M$ and $\gamma \in \mathcal{L}$, $\gamma(f)$ is not a critical point of f,*
(4) *$\mathcal{L}$ is F-invariant and $F\colon \mathcal{L} \to \mathcal{L}$ is d^k to 1.*

One of the characterization of the stability of the family $(f)_{f\in M}$ given by [Bia, Theorem C] is that this family admits an equilibrium lamination. A key step in order to obtain that all elements in M are conjugated near their small Julia set is first to construct local conjugacies near the repelling point $r(f)$.

Lemma 3.20. *Assume that M satisfies $(\star)$ and let f_0, f_1 be two points in M. For each $i \in \{0,1\}$, there exist two connected neighborhoods $\tilde{U}_i \Subset U_i$ of $r(f_i)$ with the following properties.*

- *$f_i(\tilde{U}_i) \cap U_i = \varnothing$ and f_i^2 is a biholomorphism between $\tilde{U}_i$ and U_i,*
- *there exists a biholomorphism $\psi\colon U_0 \cup f_0(\tilde{U}_0) \to U_1 \cup f_1(\tilde{U}_1)$ such that*

$$f_0 = \psi^{-1} \circ f_1 \circ \psi \quad \text{and} \quad f_0^2 = \psi^{-1} \circ f_1^2 \circ \psi \quad \text{on} \quad \tilde{U}_0, \tag{4}$$

- *if $\gamma \in \mathcal{L}$ verifies $\gamma(f_0) \in U_0$ then $\gamma(f_1) = \psi(\gamma(f_0))$,*

Observe that, since $f_0(\tilde{U}_0) \cap U_0 = \varnothing$, the first equality in (4) is simply a consequence of the definition of ψ on these sets. However, it will guarantee that ψ gives a conjugacy between f_0 and f_1 on a neighborhood of the small Julia sets as soon as the same holds between f_0^2 and f_1^2.

Proof. For $i \in \{0,1\}$, let $\mathcal{F}_{f_i}$ be the foliation of D_{f_i} defined in § 3.5. Observe that by Corollary 3.18, if $x(f_0) \in \Lambda(f_0) \cap D_{f_0}$ lies on the leaf $\mathcal{F}_{f_0}(c) := \phi_{f_0}(\pi^{-1}(c))$ then its continuation $x(f_1)$ lies on $\mathcal{F}_{f_1}(c) := \phi_{f_1}(\pi^{-1}(c))$. Replacing the family of polydisk $\Gamma(f) \subset W^u_{p(f)}$ by a similar family with $\Gamma(f) \subset \mathrm{Crit}(f)$ given by Assumption 7, we can extend this result to most point in $J_k(f_0) \cap D_{f_0}$. To be more precise, let $\gamma \in \mathcal{L}$ such that $\gamma(f_0) \in D_{f_0}$. Since sets of the form $(\Gamma_{k_n,l_n}(f_0))$ with $\Gamma(f_0) \subset \mathrm{Crit}(f_0)$ can approximate every leaf $\mathcal{F}_{f_0}(c)$, and since $\gamma(f)$ is never in the postcritical set of f, an argument of proper intersection gives that $\pi \circ \delta_f(\gamma(f))$ is independent of f.

In order to define ψ, it is sufficient to find, for $i \in \{0,1\}$, $k-1$ foliations $(\mathcal{G}^j_{f_i})_{1\le j\le k-1}$ near $r(f_i)$ which satisfy the same invariant property and such that $(\mathcal{F}_{f_i}, \mathcal{G}^1_{f_i}, \dots, \mathcal{G}^{k-1}_{f_i})$ defines local coordinates near $r(f_i)$. For this last condition, it is sufficient to check that the k tangent spaces at $r(f_i)$ of these k foliations form a family of k linearly independent hyperplanes.

To this aim, observe first that by Assumption 6, $p(f_i)$ is in the domain of linearization of $r(f_i)$ and, thus there exists $n_0 \ge 1$ such that $f_i^{n_0}$ sends biholomorphically an open subset $V_i \subset D_{f_i}$ to a neighborhood V_i' of $p(f_i)$. We denote by $v_i\colon V_i' \to V_i$ the associated inverse branch of $f_i^{n_0}$. Moreover, the cone condition in Assumption 2 ensures that the leaves of $\mathcal{G}^0_{f_i} := f_i^{n_0}(\mathcal{F}_{f_i|V_i})$ are transverse to $W^s_{p(f_i),loc}$. On the other hand, recall that (†) holds on M thus there exist $m \in \mathbb{N}$ and $x(f_i) \in W^u_{p(f_i),loc} \cap \Lambda(f_i)$ such that $f_i^m(x(f_i)) = r(f_i)$ and that, by increasing m if necessary, we can assume that f_i^m sends biholomorphically a neighborhood of $x(f_i)$ in $W^u_{p(f_i),loc}$ to a vertical graph $W_m(f_i)$ in D_{f_i}. Hence, by the inclination lemma, there exist $n_1 \ge 1$, a neighborhood $U_i \subset D_{f_i}$ of $r(f_i)$ and a small open set $V_i'' \subset V_i'$ close to $p(f_i)$ such that

- $U_i \subset f_i^2(U_i)$,
- $f_i^{n_1}\colon V_i'' \to U_i$ is a biholomorphism, whose inverse will be denoted by u_i,
- the leaves of $\mathcal{G}^1_{f_i} := f_i^{n_1}(\mathcal{G}^0_{f_i|V_i''})$ are all C^1-close to $W_m(f_i)$. In particular, the point **(2)** in Definition 3.1 implies that the tangent space of the leaf of $\mathcal{G}^1_{f_i}$ containing $r(f_i)$ is a generic hyperplane for $D_{r(f_i)}f_i^2$.

This last point and our choice of m ensure that each leaf of $\mathcal{G}^1_{f_i}$ intersects $\Lambda(f_i)$. Moreover, $\mathcal{G}^1_{f_i}$ has the same invariant property than $\mathcal{F}_{f_i}$, i.e. if, for some $\gamma \in \mathcal{L}$, $\gamma(f_0)$ is in U_0 and lies on a certain leaf of $\mathcal{G}^1_{f_0}$ then $\gamma(f_1)$ lies on the corresponding leaf of $\mathcal{G}^1_{f_1}$. To be more precise, first observe that, by reducing possibly each U_i, we can assume that $U_1 = \phi_{f_1} \circ \delta_{f_0}(U_0)$. Moreover, as in the beginning of this proof, the fact that each leaf of $\mathcal{F}_{f_i}$ can be approximated by $\Gamma_{k,l}(f_i)$ in the postcritical set of f_i and properties **(3)** and **(4)** in Definition 3.19 imply that if $\gamma(f_0) \in U_0$ then

$$\pi \circ \delta_{f_0} \circ v_0 \circ u_0(\gamma(f_0)) = \pi \circ \delta_{f_1} \circ v_1 \circ u_1(\gamma(f_1)).$$

The other foliations are simply defined as $\mathcal{G}^j_{f_i} := \left(f_i^{2(j-1)}(\mathcal{G}^1_{f_i})\right)_{|U_i}$. They also have the above invariant property since the same arguments imply

$$\pi \circ \delta_{f_0} \circ v_0 \circ u_0 \circ g_0^{j-1}(\gamma(f_0)) = \pi \circ \delta_{f_1} \circ v_1 \circ u_1 \circ g_1^{j-1}(\gamma(f_1)),$$

if $\gamma \in \mathcal{L}$ with $\gamma(f_0) \in U_0$, where g_i is the local inverse of f_i^2 near $r(f_i)$. Furthermore, the fact that the leaf of $\mathcal{G}^1_{f_i}$ containing $r(f_i)$ is a generic hyperplane for $D_{r(f_i)}f_i^2$ assures that the tangent spaces of $\mathcal{F}_{f_i}$, $\mathcal{G}^1_{f_i}, \dots, \mathcal{G}^{k-1}_{f_i}$ at $r(f_i)$ are k linearly independent hyperplanes. Hence, possibly by reducing U_i, these foliations define coordinates on U_i, i.e. since $\mathcal{F}_{f_i}$ (resp. $\mathcal{G}^j_{f_i}$) corresponds to the fibration defined by $\pi \circ \delta_{f_i}$ (resp. $\pi \circ \delta_{f_i} \circ v_i \circ u_i \circ g_i^{j-1}$), there exist an open subset $\hat{U}_i \subset \mathbb{C}^k$ such that the holomorphic map $\psi_i \colon U_i \to \hat{U}_i$ defined by

$$\psi_i(x) = \left(\pi \circ \delta_{f_i}(x), \pi \circ \delta_{f_i} \circ v_i \circ u_i(x), \cdots, \pi \circ \delta_{f_i} \circ v_i \circ u_i \circ g_i^{k-2}(x)\right)$$

is a biholomorphism.

Possibly by reducing again these sets, $\psi := \psi_1^{-1} \circ \psi_0$ is a biholomorphism between U_0 and U_1. Furthermore, the discussion above and Corollary 3.18 imply that if $\gamma \in \mathcal{L}$ verifies $\gamma(f_0) \in U_0$ then $\gamma(f_1) \in U_1$ and $\gamma(f_1) = \psi(\gamma(f_0))$. In particular, $\psi(r(f_0)) = r(f_1)$. Thus, using that these points are 2-periodic and possibly by changing a last time U_0 and U_1, we can assume that, for $i \in \{0,1\}$, there exists a connected neighborhood $\tilde{U}_i \Subset U_i$ of $r(f_i)$ such that $f_i(\tilde{U}_i) \cap U_i = \varnothing$ and f_i^2 defined a biholomorphism between $\tilde{U}_i$ and U_i. This allows us to extend ψ on $f_0(\tilde{U}_0)$ by $\psi(x) := f_1 \circ \psi(f_0^{-1}(x))$ which artificially gives

$$f_0 = \psi^{-1} \circ f_1 \circ \psi \quad \text{on} \quad \tilde{U}_0.$$

On the other hand, coming back to f_0^2 and f_1^2, the fact that $\psi(\gamma(f_0)) = \gamma(f_1)$ for each $\gamma \in \mathcal{L}$ with $\gamma(f_0) \in U_0$ implies that

$$f_0^2 = \psi^{-1} \circ f_1^2 \circ \psi \quad \text{on} \quad \tilde{U}_0 \cap \Lambda(f_0). \tag{5}$$

Since Assumption 2 guarantees the point of the blender are in the small Julia set which is not contained in an analytic subset of $\tilde{U}_0$, [FS], then $\tilde{U}_0 \cap \{\gamma(f_0) \mid \gamma \in \mathcal{L}\}$ is not contained in an analytic subset and the equality (5) holds on the whole set $\tilde{U}_0$. □

To emphasize on the dependency of ψ on f_1, in what follows we will denote ψ_f the corresponding map for $f \in M$ where f_0 stays fixed.

Lemma 3.21. *The closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ is a unbranched lamination. In particular, ψ_f extends to a conjugacy between $J_k(f_0)$ and $J_k(f)$.*

Proof. Let $f_1 \in M$ and let U_i, $i \in \{0,1\}$, be as in Lemma 3.20. Let $(\gamma_n)_{n\geq 0}$ and $(\rho_n)_{n\geq 0}$ be two sequences in $\mathcal{L}$ which converge toward two maps from M to $\mathbb{P}^k$, γ and ρ respectively. Assume also that $\gamma(f_1) = \rho(f_1)$. Our first aim is to show that $\gamma = \rho$ on M.

Let $N \geq 0$ be such that there exists $x \in f_1^{-N}(\gamma(f_1)) \cap U_1$. For $n \geq 0$ large enough, let $x_n \in U_1$ (resp. $y_n \in U_1$) be such that $f_1^N(x_n) = \gamma_n(f_1)$, $f_1^N(y_n) = \rho_n(f_1)$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x.$$

The point **(4)** in Definition 3.19 gives the existence of two sequences $(\tilde{\gamma}_n)_{n\geq 0}$ and $(\tilde{\rho}_n)_{n\geq 0}$ in $\mathcal{L}$ such that $F^N(\tilde{\gamma}_n) = \gamma_n$, $F^N(\tilde{\rho}_n) = \rho_n$ and $\tilde{\gamma}_n(f_1) = x_n$, $\tilde{\rho}_n(f_1) = y_n$. Up to a subsequence, we can assume that $(\tilde{\gamma}_n)_{n\geq 0}$ and $(\tilde{\rho}_n)_{n\geq 0}$ converge to two maps $\tilde{\gamma}$ and $\tilde{\rho}$. Since by Lemma 3.20 $\psi_{f_1}(\tilde{\gamma}_n(f_0)) = \tilde{\gamma}_n(f_1)$ and $\psi_{f_1}(\tilde{\rho}_n(f_0)) = \tilde{\rho}_n(f_1)$ we have $\psi_{f_1}(\tilde{\gamma}(f_0)) = \psi_{f_1}(\tilde{\rho}(f_0))$ and thus $\tilde{\gamma}(f_0) = \tilde{\rho}(f_0)$ by injectivity of ψ_{f_1}. By applying F^N, we also have $\gamma(f_0) = \rho(f_0)$. The same arguments with an arbitrary map $f \in M$ give $\gamma = \rho$. This proves that $\overline{\mathcal{L}}$ is unbranched.

From this, we can defined, for every $\gamma \in \overline{\mathcal{L}}$, $\psi_f(\gamma(f_0)) := \gamma(f)$. Since $\overline{\mathcal{L}}$ is unbranched, it extends ψ_f as a conjugacy between $J_k(f_0)$ and $J_k(f)$. □

The extension of ψ_f to a neighborhood of $J_k(f_0)$ comes from the following partial generalization of [BE1] to higher dimensions.

Proposition 3.22. *Let f_0 and f_1 be two endomorphisms of $\mathbb{P}^k$ of degree $d \geq 2$. Assume there exist an open set V_0 and a continuous map $\psi\colon V_0 \cup J_k(f_0) \to \mathbb{P}^k$ such that*

- *$\psi_{|J_k(f_0)}$ is a homeomorphism from $J_k(f_0)$ to $J_k(f_1)$ such that $\psi \circ f_0 = f_1 \circ \psi$ on $J_k(f_0)$,*
- *$V_0 \cap J_k(f_0) \neq \varnothing$ and $\psi_{|V_0}$ is holomorphic.*

Assume also that the exceptional set $\mathcal{E}(f_0)$ of f_0 is disjoint from $J_k(f_0)$. Then, there exist two open neighborhoods $N_1 \subset N_2$ of $J_k(f_0)$ such that

- *$f_0(N_1) \subset N_2$,*
- *ψ extends to a holomorphic map $\tilde{\psi}$ on N_2 such that $f_1 \circ \tilde{\psi} = \tilde{\psi} \circ f_0$ on N_1.*

For the proof, let C be the critical set of f_0, $A := f_0(C)$ its set of critical values and $B := f_0^{-1}(A)$.

Lemma 3.23. *Let $x \in \mathbb{P}^k \setminus B$, and let $\gamma'\colon [0,1] \to \mathbb{P}^k \setminus A$ be a path such that $\gamma'(0) = f(x)$. Then there exists a unique path $\gamma\colon [0,1] \to \mathbb{P}^k \setminus B$ such that $\gamma(0) = x$ and $f_0(\gamma(t)) = \gamma'(t)$ for all $t \in [0,1]$.*

Proof. This simply comes from the fact that $f_0\colon \mathbb{P}^k \setminus B \to \mathbb{P}^k \setminus A$ is a covering map. □

Lemma 3.24. *Let $x \in \mathbb{P}^k$ and let V be a neighborhood of x. There exists a connected open neighborhood V_x (resp. W_x) of x (resp. of $f_0(x)$) such that*

- *$V_x \subset V$ and $W_x \subset f_0(V_x)$,*
- *if $\gamma'\colon [0,1] \to W_x \setminus A$ is a path and $y \in f_0^{-1}(\gamma'(0)) \cap V_x$ then there exists a path γ in V_x with $\gamma(0) = y$ and $f_0(\gamma(t)) = \gamma'(t)$ for all $t \in [0,1]$.*

Notice that in the following proof we use a Lojasiewicz type inequality but the fact that f is finite and open is sufficient.

Proof. Let $V_x \subset V$ be a connected open neighborhood of x such that $f_0^{-1}(f_0(x)) \cap \partial V_x = \varnothing$, i.e. $\mathrm{dist}(\partial V_x, f_0^{-1}(f_0(x))) = a$ with $a > 0$. A Lojasiewicz type inequality ([FS, Corollary 4.12] when $k = 2$) gives that there exists a constant $c > 0$, depending only on f_0, such that

$$\mathrm{dist}(f_0(\partial V_x), f_0(x)) \geq c a^{d^k}.$$

Since f_0 is an open mapping, there exists $\epsilon > 0$ such that $\epsilon < ca^{d^k}$ and $W_x := B(f_0(x), \epsilon) \subset f_0(V_x)$. In particular, $f_0(\partial V_x) \cap W_x = \varnothing$. Hence, if $\gamma' \colon [0,1] \to W_x \setminus A$ is a path and $y \in f_0^{-1}(\gamma'(0)) \cap V_x$ then by Lemma 3.23 we can lift γ' to a path γ in $\mathbb{P}^k$ such that $\gamma(0) = y$. And, since $\gamma'([0,1]) \subset W_x$ and $f_0(\partial V_x) \cap W_x = \varnothing$, we must have $\gamma([0,1]) \subset V_x$. □

Lemma 3.25. *The set of points x in $J_k(f_0)$ where ψ admits a holomorphic extension in a neighborhood of x is f_0-invariant.*

Proof. Let $x \in J_k(f_0)$ be such a point and let V be a neighborhood of x where ψ extends holomorphically. The interesting case is when x is a critical point of f_0. Let V_x and W_x the connected open neighborhood of x and $f_0(x)$ respectively given by Lemma 3.24. Let $y \in W_x \setminus A$ and let y_1 and y_2 be two points in $\in f_0^{-1}(y) \cap V_x$. The goal is to show that $f_1(\psi(y_1)) = f_1(\psi(y_2))$ since in that case, we can define ψ on $W_x \setminus A$ using local inverse branches of f_0 and the definition will extends to W_x.

Since x is in $J_k(f_0)$ the same holds for $f_0(x)$. The fact that $J_k(f_0)$ is nowhere pluripolar [FS] implies the existence of $z \in (W_x \cap J_k(f_0)) \setminus A$. Let γ' be a simple path in $W_x \setminus A$ between y and z. By Lemma 3.24, it admits two lifts γ_1 and γ_2 in V_x such that $\gamma_1(0) = y_1$ and $\gamma_2(0) = y_2$. The end points $z_1 := \gamma_1(1)$ and $z_2 := \gamma_2(1)$ are preimages of z and thus are in $J_k(f_0)$. Since γ' is simple, by analytic continuation there exist a connected neighborhood Ω of $\gamma'([0,1])$ and two holomorphic maps, g_1 and g_2, from Ω to V_x such that for $i \in \{1,2\}$,

- g_i is an inverse branch of f_0, i.e. $f_0 \circ g_i = \mathrm{id}_\Omega$,
- $\gamma_i(t) = g_i(\gamma'(t))$ for all $t \in [0,1]$.

Since $\psi_{|J_k(f_0)}$ conjugates f_0 to f_1 on $J_k(f_0)$, we have $f_1 \circ \psi \circ g_i = \psi$ on $\Omega \cap J_k(f_0)$ for $i \in \{1,2\}$. Hence, the fact that $\Omega \cap J_k(f_0)$ is not pluripolar and the connectedness of Ω implies that $f_1 \circ \psi \circ g_1 = f_1 \circ \psi \circ g_2$ on Ω. In particular $f_1(\psi(y_1)) = f_1(\psi(g_1(y))) = f_1(\psi(g_2(y))) = f_1(\psi(y_2))$. □

From this, the proof of Proposition 3.22 is identical to the one of [BE1, Lemma 3] but we give it for completeness.

Proof of Proposition 3.22. Since $J_k(f_0) \cap \mathcal{E}(f_0) = \varnothing$, there exists $N \geq 1$ such that $J_k(f_0) \subset f_0^N(U_1)$. Hence, Lemma 3.25 implies that for all $x \in J_k(f_0)$ there is a holomorphic extension ψ_x of ψ in a neighborhood of x. In particular, there exists $r_x > 0$ such that ψ_x is defined on $B(x, 3r_x)$. Observe that if $x, y \in J_k(f_0)$ are such that $r_x \leq r_y$ and $B(x, r_x) \cap B(y, r_y) \neq \varnothing$ then $B(x, r_x) \subset B(y, 3r_y)$. In particular, by non-pluripolarity of $B(x, r_x) \cap J_k(f_0)$, we have that $\psi_x = \psi_y$ on $B(x, r_x)$. Hence, ψ has a holomorphic extension $\tilde{\psi}$ on

$$N_2 := \bigcup_{x \in J_k(f_0)} B(x, r_x).$$

By continuity of f_0, there exists an open neighborhood $N_1 \subset N_2$ of $J_k(f_0)$ such that $f_0(N_1) \subset N_2$. We also can assume that each connected component of N_1 intersects $J_k(f_0)$. If N is such connected component then $f_1 \circ \tilde{\psi} = \tilde{\psi} \circ f_0$ on $N \cap J_k(f_0)$ by definition of ψ and thus by analytic continuation $f_1 \circ \tilde{\psi} = \tilde{\psi} \circ f_0$ on N. □

This allows us to finish to proof of Theorem 3.4.

Proof of Theorem 3.4. Let M be an analytic subset of Ω satisfying $(\star)$. Let f_0 and f_1 be two elements of M. By Lemma 3.20 and Lemma 3.21, there exists a map ψ, given on $J_k(f_0)$ by the unbranched holomorphic motion $\overline{\mathcal{L}}$, which verifies the assumption of

Proposition 3.22. Observe that we use here Assumption 9 which ensures that $J_k(f_0)$ is disjoint from the exceptional set $\mathcal{E}(f_0)$. Hence, there are two neighborhood $N_1 \subset N_2$ of $J_k(f_0)$ and a holomorphic map $\tilde{\psi}$ on N_2 such that $f_1 \circ \tilde{\psi} = \tilde{\psi} \circ f_0$ on N_1. This directly implies that all the periodic points in $J_k(f_0)$ can be followed holomorphically on M and that their multipliers are constant on M. □

4. Existence of a good open subset in End_d^k

The aim of this section is to prove the following existence statement.

Theorem 4.1. *There exist $(a, \epsilon, \sigma_3, \dots, \sigma_k) \in (\mathbb{R}_{>0})^k$ and a small perturbation $f \in \mathrm{Poly}_d^k$ of the map $f_0 : \mathbb{C}^k \to \mathbb{C}^k$ given by*

$$f_0(z, w, y_3, \dots, y_k) = (e^{i\pi/4} z + \epsilon w, a(w^2 - 1), \sigma_3 y_3, \dots, \sigma_k y_k)$$

such that f admits a neighborhood Ω in End_d^k which satisfies all the assumptions described in § 3.2.

Remark 4.2. Since the map f in Theorem 4.1 belongs to Poly_d^k, the result also provides a non-empty open subset of Poly_d^k satisfying all the assumptions of § 3.2.

The structure of the section is the following. In § 4.1, we recall elementary results about the dynamics of $w \mapsto a(w^2 - 1)$ when $|a|$ is large. In § 4.2, we start to study the 2-dimensional case which is the most important one. In particular, Lemma 4.5 and Lemma 4.6 settle the blender property for the repelling hyperbolic set Λ. Observe that for $d = 2$, it is delicate to obtain a saddle point and a repelling hyperbolic set which form a heterodimensional cycle. This explains why we have to work with the second or the fourth iterates of our maps. § 4.2 is also devoted to the study of the degeneracy of these maps when the parameter a goes to infinity. This is the key ingredient to check Assumption 10 of § 3.2, which is by far the more difficult to obtain. The case of higher dimension is considered in § 4.3 where the parameters are chosen more carefully, in particular to linearize in family the dynamics near the two periodic points p and r. § 4.4 is devoted to establish the point iii) in Assumption 10. Finally, we prove Theorem 4.1 in § 4.5.

4.1. Dynamics of $a(w^2 - 1)$.

For $a \in \mathbb{C}^*$ we consider the polynomial map $q_a(w) := a(w^2 - 1)$. For $|a|$ large enough, q_a is hyperbolic with a Cantor set as Julia set. In what follows, we will construct a blender for a map of the form $(z, w) \mapsto (g(z, w), q_a^4(w))$. To this end, we need to consider open sets with a specific (but simple) combinatoric.

From now on, we fix $a \in \mathbb{C}^*$ with $|a| > 10$.

Lemma 4.3. *There exists a neighborhood U_+ (resp. U_-) of 1 (resp. -1) such that $q_{a|U_\pm}$ is a biholomorphism between $U_\pm$ and $\mathbb{D}_3$, and*

$$D(\pm 1, |a|^{-1}) \subset U_\pm \subset D(\pm 1, 3|a|^{-1}).$$

In particular, q_a admits a unique fixed point $\tilde{w}(a) \in U_-$.

This implies that the Julia set of q_a is a Cantor set, equal to $J_{q_a} = \cap_{n \geq 0} q_a^{-n}(U_+ \cup U_-)$. Its dynamics corresponds to a (one-sided) full 2-shift. However, our construction will not use the entire J_{q_a} but two smaller hyperbolic sets.

The first one is simply the unique fixed point $\tilde{w}(a) \in U_-$. For the second one, let $g_+ \colon \mathbb{D}_3 \to U_+$ and $g_- \colon \mathbb{D}_3 \to U_-$ be the two inverse branches of q_a obtained in Lemma 4.3. From this, we define the following open sets:

- $V_- := g_-(U_+)$ and $V_+ := g_+(U_-)$,

- $V_1 := q_a^{-2}(V_+)\cap V_+$, $V_2 := q_a^{-2}(V_-)\cap V_+$, $V_3 := q_a^{-2}(V_+)\cap V_-$ and $V_4 := q_a^{-2}(V_-)\cap V_-$.

They satisfy $q_a^2(V_-) = q_a^2(V_+) = \mathbb{D}_3$ and $q_a^4(V_i) = \mathbb{D}_3$ for $i \in \{1,2,3,4\}$. The definition of V_i ensures that the associated maximal invariant sets are equal, i.e.

$$E := \bigcap_{n\geq 0} q_a^{-2n}(V_+ \cup V_-) = \bigcap_{n\geq 0} q_a^{-4n}(\cup_{i=1}^4 V_i). \tag{6}$$

It is also a Cantor set where q_a^2 is conjugated to a full 2-shift. Observe that if $\{+,-\}$ is the alphabet for $q_{a|J_{q_a}}$ then $q^2_{a|E}$ corresponds to the 2-shift associated to $\{+-,-+\}$. This alternation will play an important role in the proof of Theorem 4.1. In particular, the second coordinates of r in Assumption 6 will be the point $w_0(a)$ which is the unique 2-periodic point of q_a in $V_4 \subset V_-$. We will need the following simple fact later.

Lemma 4.4. *If a is real then the 2-periodic point $w_0(a) \in V_-$ is also real. Actually, the two inverse branches, $g_+ \colon \mathbb{D}_3 \to U_+$ and $g_- \colon \mathbb{D}_3 \to U_-$ are real. In particular, $w_l(a) := (g_+ \circ g_-)^l(w_0(a))$, are real for all $l \geq 0$.*

All the subsets defined in this section depend on a. If necessary, we will write $U_\pm(a)$, $V_\pm(a)$, $V_i(a)$ or $E(a)$ to emphasize on these dependencies.

4.2. **Perturbations of product maps and the IFS at infinity.** The construction in Theorem 4.1 starts from a skew product

$$\mathcal{F}_\lambda(z,w) = (\alpha z + \epsilon w + \beta z w, q_a(w)),$$

where $\lambda = (a,\alpha,\beta,\epsilon)$. Such a map does not extend to $\mathbb{P}^2$. The case of End_d^k with a general $k \geq 2$ will be considered in § 4.3. Several objects denoted with stylized uppercase letters in this section (e.g. $\mathcal{F}_\lambda$, $\mathcal{P}(\lambda)$, $\mathcal{R}(\lambda)$) will corresponds to lowercase letters in § 4.3 (e.g. f_λ, $p(\lambda)$, $r(\lambda)$).

A first observation is that $\mathcal{F}_{(a,\alpha,\beta,\epsilon)}$ and $\mathcal{F}_{(a,\alpha,\beta,\epsilon')}$ are globally conjugated if $\epsilon \neq 0 \neq \epsilon'$. The role of the parameter $\epsilon \neq 0$ is just to rescale the dynamics in order to have the blender property above $\mathbb{D}$.

If $\tilde{w}(a)$ denotes the unique fixed point of q_a in $U_-(a)$ then $\mathcal{F}_\lambda$ has a fixed point

$$\mathcal{P}(\lambda) = \left(\frac{-\epsilon\tilde{w}(a)}{\alpha + \beta\tilde{w}(a) - 1}, \tilde{w}(a)\right),$$

which is repelling in a vertical direction and whose multiplier in the horizontal direction is $\alpha + \beta\tilde{w}(a)$, very close to $\alpha - \beta$ when $|a|$ is large. On the other hand, by the choice of the sets $V_+(a)$ and $V_-(a)$, the dynamics in the horizontal direction of the second iterate $\mathcal{F}_\lambda^2$ is mainly a dilatation of factor $\alpha^2 - \beta^2$ on $\mathbb{C} \times (V_+(a) \cup V_-(a))$. Hence, in what follows we will choose α and β in order to have $|\alpha - \beta| < 1$, which implies that $\mathcal{P}(\lambda)$ is saddle, and $|\alpha^2 - \beta^2| > 1$ which ensures the existence of a repelling hyperbolic set $\Lambda(\lambda)$ for $\mathcal{F}_\lambda^2$. This hyperbolic set will have a blender property if α, β and ϵ are well chosen and it will project on $E(a)$. In order to check transversality properties, we will make a goes to infinity. In this situation, the set $E(a)$ degenerates to $\{-1,1\}$ and the dynamics on $\Lambda(\lambda)$ degenerates to (the inverse of) an iterated function system (IFS) with 2 generators.

To be more precise, the second iterate of $\mathcal{F}_\lambda$ is

$$\mathcal{F}_\lambda^2(z,w) = (z(\alpha^2 + \alpha\beta(w + q_a(w)) + \beta^2 w q_a(w)) + \epsilon(\alpha w + q_a(w) + \beta w q_a(w)), q_a^2(w)).$$

In particular, since

$$V_+(a) \subset q_a(V_-) = U_+(a) \subset D(1, 3|a|^{-1}) \quad \text{and} \quad V_-(a) \subset q_a(V_+) = U_-(a) \subset D(-1, 3|a|^{-1}),$$

if $\hat{\lambda} := (\alpha, \beta, \epsilon) \in \mathbb{C}^3$ and $R > 0$ are fixed then for $|a| > 10$ large enough, $\mathcal{F}^2_\lambda(z, w)$ is arbitrarily close to $(\phi^+_{\hat{\lambda}}(z), q^2_a(w))$ (resp. $(\phi^-_{\hat{\lambda}}(z), q^2_a(w))$) on $\mathbb{D}_R \times V_+(a)$ (resp. on $\mathbb{D}_R \times V_-(a)$) where

$$\phi^+_{\hat{\lambda}}(z) = (\alpha^2 - \beta^2)z - \epsilon(\beta + 1 - \alpha), \quad \phi^-_{\hat{\lambda}}(z) = (\alpha^2 - \beta^2)z - \epsilon(\beta + \alpha - 1).$$

From this, we define

$$\phi_{1,\hat{\lambda}} := \phi^+_{\hat{\lambda}} \circ \phi^+_{\hat{\lambda}},\ \phi_{2,\hat{\lambda}} := \phi^-_{\hat{\lambda}} \circ \phi^+_{\hat{\lambda}},\ \phi_{3,\hat{\lambda}} := \phi^+_{\hat{\lambda}} \circ \phi^-_{\hat{\lambda}} \ \text{ and } \ \phi_{4,\hat{\lambda}} := \phi^-_{\hat{\lambda}} \circ \phi^-_{\hat{\lambda}}$$

in order to have $\mathcal{F}^4_\lambda(z, w) \simeq (\phi_{j,\hat{\lambda}}(z), q^4_a(w))$ on $\mathbb{D}_R \times V_j(a)$.

Now, we fix a small real $A > 0$ and take $\alpha_0 := \zeta(1 + A)$ and $\beta_0 := 2A\zeta$ where $\zeta \in \mathbb{S}^1$. This gives $\alpha_0 - \beta_0 = \zeta(1 - A)$ and thus the fixed point $\mathcal{P}(a, \alpha_0, \beta_0, \epsilon)$ is saddle for $|a|$ large enough. On the other hand, $\alpha_0^2 - \beta_0^2 = \zeta^2(1 + 2A - 3A^2)$ which has modulus larger than 1 if $A < 2/3$. For the constant $\zeta \in \mathbb{S}^1$, following [Du, Lemma 4.4], we will take $\zeta = e^{i\pi/4}$ in order to have a blender property for $\Lambda(\lambda)$. The following result can be seen as the counterpart in our context of this lemma using the vocabulary of [T].

Lemma 4.5. *Let $\zeta := e^{i\pi/4}$ and $\epsilon_0 = (20(\zeta - 1))^{-1}$. Let $0 < A < 1/10$ be small enough and let set $\hat{\lambda}_0 := (\alpha_0, \beta_0, \epsilon_0)$ where*

$$\alpha_0 = \zeta(1 + A) \quad \textit{and} \quad \beta_0 = 2A\zeta.$$

Then, there exist four open sets H_j, $j \in \{1, 2, 3, 4\}$ such that

$$\mathbb{D}_2 = \cup_{j=1}^4 H_j,\ \overline{\phi_{j,\hat{\lambda}_0}(H_j)} \subset \mathbb{D}_2 \quad \textit{and } \overline{\mathbb{D}} \subset H_j.$$

Proof. For $\hat{\lambda}_1 := (\zeta, 0, \epsilon_0)$, an easy computation gives

$$\phi_{1,\hat{\lambda}_1}(z) = -z + \frac{1+i}{20},\ \phi_{2,\hat{\lambda}_1}(z) = -z + \frac{i-1}{20},\ \phi_{3,\hat{\lambda}_1}(z) = -z + \frac{1-i}{20} \text{ and } \phi_{4,\hat{\lambda}_1}(z) = -z - \frac{1+i}{20}.$$

Moreover, if we define

$$H_j = \mathbb{D}_{4/3} \cup \{z \in \mathbb{D}_2\ ;\ |\arg(z\zeta^{-2j+1})| < \pi/3\},$$

then $\overline{\phi_{j,\hat{\lambda}_1}(H_j)} \subset \mathbb{D}_2$. Actually, this simply comes from the inequalities $|2e^{i\pi/3} - \sqrt{2}/20| < 2$ and $|\sqrt{2}/20| < 1$. Since this inclusion is stable under small perturbations, if $A > 0$ is small enough and $j \in \{1, 2, 3, 4\}$ then $\overline{\phi_{j,\hat{\lambda}_0}(H_j)} \subset \mathbb{D}_2$ when $\hat{\lambda}_0 = (\zeta(1 + A), 2A\zeta, \epsilon_0)$. On the other hand, $\mathbb{D}_2 = \cup_{j=1}^4 H_j$ and $\overline{\mathbb{D}} \subset \cap_{j=1}^4 H_j$ easily follow from the definition of H_j. □

Since the properties in Lemma 4.5 are stable under small perturbations, they persist in a small neighborhood of $\hat{\lambda}_0 = (\alpha_0, \beta_0, \epsilon_0)$. From now on, we denote by $\hat{M}$ such small neighborhood of $\hat{\lambda}_0$ which is connected and where, moreover, for all $\hat{\lambda} = (\alpha, \beta, \epsilon) \in \hat{M}$

$$1/20 < |\epsilon| < 1/10,\ |\alpha - \beta| < 1 \text{ and } |\alpha^2| - |\beta^2| > 1 + A \tag{7}$$

In particular, if $R := A^{-1}$, the maps $\phi^\pm_{\hat{\lambda}}$ satisfies $\overline{\mathbb{D}_R} \subset \phi^\pm_{\hat{\lambda}}(\mathbb{D}_R)$.

The next step is to define the hyperbolic set Λ with the blender property which appears in Assumptions 2 and 4.

Lemma 4.6. *There exist $\rho > 100$ and $\delta > 100$ such that, if $|a| > \delta$ then for every $(\alpha, \beta, \epsilon) \in \hat{M}$ the map $\mathcal{F}_\lambda$, with $\lambda := (a, \alpha, \beta, \epsilon)$, satisfies the following properties.*

- *On both $\mathbb{D}_R \times V_-(a)$ and $\mathbb{D}_R \times V_+(a)$ the map $\mathcal{F}_\lambda^2$ is injective, contracts the cone field C_ρ and is expanding. Moreover*

$$\overline{\mathbb{D}_R \times (U_+(a) \cup U_-(a))} \subset \mathcal{F}_\lambda^2(\mathbb{D}_R \times V_\pm(a)).$$

  In particular, the set $\Lambda(\lambda) := \cap_{n\geq 0}\mathcal{F}_\lambda^{-2n}(\mathbb{D}_R \times (V_-(a) \cup V_+(a)))$ is a hyperbolic repelling invariant set for $\mathcal{F}_\lambda^2$.
- *For $i \in \{1,2,3,4\}$, any vertical graph in $H_i \times V_i(a)$ tangent to the cone C_ρ intersects $\Lambda(\lambda)$.*

Moreover, both statements are stable under small C^1-perturbations of $\mathcal{F}_\lambda$.

Proof. Let $\hat{\lambda} = (\alpha, \beta, \epsilon)$ be in $\hat{M}$. The key ingredient is that if $|a|$ is large enough then $\mathcal{F}_\lambda^2$ is arbitrarily close to the product map $(\phi_{\hat{\lambda}}^\pm, q_a^2)$ on $\mathbb{D}_R \times V_\pm(a)$ and $\mathcal{F}_\lambda^4$ is arbitrarily close to $(\phi_{j,\hat{\lambda}}, q_a^4)$ on $H_j \times V_j(a)$. This gives easily that $\mathcal{F}_\lambda^2$ is expanding on $\mathbb{D}_R \times V_\pm(a)$ and also injective since q_a^2 is injective on $V_\pm(a)$. Moreover, $\mathcal{F}_\lambda^2$ contracts the cone field C_ρ on $\mathbb{D}_R \times V_\pm(a)$ for $|a|$ large since the derivative of $\mathcal{F}_\lambda^2$ in the vertical direction is bounded from below by $|a|^2$ while the derivative in the horizontal direction is uniformly bounded from above on $\mathbb{D}_R \times V_\pm(a)$. Hence, for every $\rho > 0$ there exists $\delta > 0$ such that $\mathcal{F}_\lambda^2$ contracts the cone field C_ρ on $\mathbb{D}_R \times V_\pm(a)$ as soon as $|a| > \delta$.

We also have $\overline{\mathbb{D}_R \times (U_+(a) \cup U_-(a))} \subset \mathcal{F}_\lambda^2(\mathbb{D}_R \times V_\pm(a))$ since $\overline{U_+(a) \cup U_-(a)} \subset \mathbb{D}_3 = q_a^2(V_\pm(a))$ and $\overline{\mathbb{D}_R} \subset \phi_{\hat{\lambda}}^\pm(\mathbb{D}_R)$. From this, it is classical that $\Lambda(\lambda) := \cap_{n\geq 0}\mathcal{F}_\lambda^{-2n}(\mathbb{D}_R \times (V_-(a) \cup V_+(a)))$ is a hyperbolic repelling set for $\mathcal{F}_\lambda^2$. Actually, it is easy to see that $\Lambda(\lambda)$ is homeomorphic, via the second canonical projection, to the corresponding set $E(a)$ for q_a^2 defined by (6) which is a Cantor set. Observe that, as in (6), we also have $\Lambda(\lambda) = \cap_{n\geq 0}\mathcal{F}_\lambda^{-4n}(\mathbb{D}_R \times (\cup_{j=1}^4 V_j(a)))$.

The second statement is the counterpart of [Du, Lemma 4.5] or [T, Proposition 3.3] in our setting and we only sketch the proof. Let H_j, $j \in \{1,2,3,4\}$, be the four open subsets of $\mathbb{D}_2$ defined in Lemma 4.5. Recall that $\mathbb{D}_2 = \cup_{j=1}^4 H_j$, $\overline{\mathbb{D}} \subset H_j$ and $\overline{\phi_{j,\hat{\lambda}}(H_j)} \subset \mathbb{D}_2$, where this last inclusion comes from our choice of $\hat{M}$. In particular, there exists $r > 0$ such that $\cup_{j=1}^4 \overline{\phi_{j,\hat{\lambda}}(H_j)} \subset \mathbb{D}_{2-r}$ and thus, if $|a|$ is large enough and we set $\mathcal{F}_\lambda^4(z,w) = (F_\lambda(z,w), q_a^4(w))$ then

$$\bigcup_{j=1}^4 \overline{F_\lambda(H_j \times V_j(a))} \subset \bigcup_{j=1}^4 H_j.$$

Let $\eta > 0$ denote the Lebesgue number of this open cover. If $\rho > 0$ is large enough then the projection on the first coordinate of a vertical graph tangent to C_ρ has diameter less than η. Hence, if $j_0 \in \{1,2,3,4\}$ and $\Gamma_0 \subset H_{j_0} \times V_{j_0}(a)$ is a vertical graph tangent to C_ρ then $\mathcal{F}_\lambda^4(\Gamma_0)$ contains a vertical graph Γ_1 in $H_{j_1} \times V_{j_1}(a)$ for some $j_1 \in \{1,2,3,4\}$ which is tangent to C_ρ. By induction, we obtain a sequence of vertical graph $\Gamma_n \subset \mathcal{F}_\lambda^{4n}(\Gamma_0)$ in some $H_{j_n} \times V_{j_n}(a)$ and thus Γ_0 intersects $\cap_{n\geq 0}\mathcal{F}_\lambda^{-4n}(\mathbb{D}_R \times (\cup_{j=1}^4 V_j(a))) = \Lambda(\lambda)$. □

By construction, each point in $\Lambda(\lambda)$ is associated to a unique word ω in $\Sigma := \{-1,1\}^{\mathbb{N}}$. To be more precise, let $g_{\lambda,+} \colon \mathbb{D}_R \times \mathbb{D}_2 \to \mathbb{D}_R \times V_+(a)$ and $g_{\lambda,+} \colon \mathbb{D}_R \times \mathbb{D}_2 \to \mathbb{D}_R \times V_-(a)$ be the two inverse branches of $\mathcal{F}_\lambda^2$. If we identify the symbol $+$ with 1 and $-$ with -1 then a word $\omega = (\omega_n)_{n\geq 0} \in \Sigma := \{-1,1\}^{\mathbb{N}}$ induces a dynamical system $(g_{\lambda,\omega}^n)_{n\geq 0}$ where

$$g_{\lambda,\omega}^n := g_{\lambda,\omega_0} \circ \cdots \circ g_{\lambda,\omega_{n-1}}.$$

Since the maps $g_{\lambda,\pm}$ are contracting, the sequence $(g^n_{\lambda,\omega}(y))_{n\geq 0}$ has a limit, denoted by $x_\omega(\lambda)$, which is independent of $y \in \mathbb{D}_R \times \mathbb{D}_2$. The hyperbolic set $\Lambda(\lambda)$ corresponds exactly to $\{x_\omega(\lambda)\ ;\ \omega \in \Sigma\}$ and the repelling point $\mathcal{R}(\lambda)$ in Assumption 6 is, in this situation, $x_\omega(\lambda)$ where $\omega = (-1)_{n\geq 0}$. Observe that $x_\omega(\lambda)$ depends holomorphically on λ and continuously on ω with respect to the product topology on Σ.

In order to check Assumption 10, we are interested in parameters $\lambda = (a, \alpha, \beta, \epsilon)$ and $\omega \in \Sigma$ where $x_\omega(\lambda)$ is a preimage of $\mathcal{R}(\lambda)$ lying on $W^u_{\mathcal{P}(\lambda),loc}$ with a additional condition on the multipliers. Here, we consider the local unstable manifold in $\mathbb{D}_R \times D(-1, 1/2)$, which is a vertical graph since one easily check that $\mathcal{F}_\lambda$ contracts C_ρ there. In what follows, we will study the degeneracy of these relations $x_\omega(\lambda) \in W^u_{\mathcal{P}(\lambda),loc}$ when α, β and ϵ are fixed and $|a|$ tends to infinity. In this situation, the maps $g_{\lambda,\pm}$ converge to $(\ell_{\hat\lambda,\pm}, \pm 1)$ where

$$\ell_{\hat\lambda,+}(z) = \mu z + \nu_+ \ \text{ and } \ \ell_{\hat\lambda,-}(z) = \mu z + \nu_-, \tag{8}$$

with $\hat\lambda = (\alpha, \beta, \epsilon)$, $\mu := (\alpha^2 - \beta^2)^{-1}$ and $\nu_\pm := \mu\epsilon(\beta \pm (1-\alpha))$. Hence, when $|a|$ goes to infinity, the first projection of $\Lambda(\hat\lambda, a)$ degenerates to the limit set of the IFS $\mathcal{L}_{\hat\lambda}$ generated by $\ell_{\hat\lambda,+}$ and $\ell_{\hat\lambda,-}$. The point $x_\omega(\lambda)$ converges to $(z_\omega(\hat\lambda), \omega_0)$ where

$$z_\omega(\hat\lambda) := \lim_{n\to\infty} \ell_{\hat\lambda,\omega_0} \circ \cdots \circ \ell_{\hat\lambda,\omega_{n-1}}(y)$$

for any $y \in \mathbb{C}$. The limit set of $\mathcal{L}_{\hat\lambda}$ is then equal to $\{z_\omega(\hat\lambda)\ ;\ \omega \in \Sigma\}$.

The definition of $\ell_\pm$ in (8) ensures that

$$z_\omega(\hat\lambda) = \sum_{n\geq 0} \nu_{\omega_n} \mu^n.$$

Using the definition of $\nu_\pm$, this gives

$$z_\omega(\hat\lambda) = \epsilon\mu \left(\frac{\beta}{1-\mu} + (1-\alpha) h_\omega(\mu) \right), \tag{9}$$

where $h_\omega(\mu) := \sum_{n\geq 0} \omega_n \mu^n$. On the other hand, $\mathcal{F}_\lambda$ contracts C_ρ on $\mathbb{D}_R \times D(-1, 1/2)$ thus when a goes to infinity, the unstable manifold $W^u_{\mathcal{P}(a,\alpha,\beta,\epsilon),loc}$ converges to the vertical line $W^u_{\mathcal{P}(\infty,\alpha,\beta,\epsilon),loc} := \{z = \frac{\epsilon}{\alpha-\beta-1}\}$. These information help us to understand the relation $x_\omega(\lambda) \in W^u_{\mathcal{P}(\lambda),loc}$. The following result will in particular imply Assumption 5. As a will converge to infinity, we denote by $D(\infty, r)$ the set $\{\infty\} \cup \{y \in \mathbb{C}\ ;\ |y| > 1/r\}$.

Lemma 4.7. *Let $\delta > 100$ be as in Lemma 4.6 and $(\alpha_0, \beta_0, \epsilon_0)$ be as in Lemma 4.5. There exists a non-empty connected open neighborhood M of $(\infty, \alpha_0, \beta_0, \epsilon_0)$ in $D(\infty, 1/\delta) \times \hat M$ such that for each $\omega \in \Sigma$, the analytic set*

$$X_\omega := \left\{ \lambda \in D(\infty, 1/\delta) \times \hat M\ ;\ x_\omega(\lambda) \in W^u_{\mathcal{P}(\lambda),loc} \right\}$$

is a (possibly empty) hypersurface and each irreducible component of X_ω that intersects M also intersects $\{\infty\} \times \hat M$.

Proof. It is clear that X_ω is an analytic set of codimension at most 1 (if not empty). Let assume that for some $\omega \in \Sigma$ we have $X_\omega = D(\infty, 1/\delta) \times \hat M$. In particular, with $a = \infty$ we have for all $(\alpha, \beta, \epsilon) \in \hat M$, and after simplification by ϵ, that

$$\mu \left(\frac{\beta}{1-\mu} + (1-\alpha) h_\omega(\mu) \right) = \frac{1}{\alpha - \beta - 1},$$

where $\mu = (\alpha^2 - \beta^2)^{-1}$. As $\hat{M}$ is open and as the radius of convergence of h_ω is 1, this equality should hold for all $(\alpha, \beta) \in \mathbb{C}^2$ with $|\alpha^2 - \beta^2| > 1$ which is impossible with $\alpha = 2$ and $\beta = 1$ since the right hand side diverges.

Observe that we have proved the stronger result that the intersections between each X_ω and $\{\infty\} \times \hat{M}$ are proper. This will allow us to prove the second statement by contradiction. Assume there exist a sequence $(\omega_n)_{n\geq 0}$ in Σ and a sequence $(\lambda_n)_{n\geq 0}$ in $\hat{M}$ converging toward $\lambda_\infty := (\infty, \alpha_0, \beta_0, \epsilon_0)$ such that λ_n belongs to an irreducible component C_{ω_n} of X_{ω_n} which is disjoint from $\{\infty\} \times \hat{M}$. Up to a subsequence, $(\omega_n)_{n\geq 0}$ converges to some $\omega_\infty \in \Sigma$ and thus X_{ω_n} converges to X_{ω_∞} and C_{ω_n} converges to a union of irreducible components C_{ω_∞} of X_{ω_∞}. Since $\lambda_n \in C_{\omega_n}$, we must have $\lambda_\infty \in C_{\omega_\infty}$, i.e. $C_{\omega_\infty} \cap \{\infty\} \times \hat{M} \neq \varnothing$. The fact that C_{ω_n} is disjoint from $\{\infty\} \times \hat{M}$ then contradicts the persistence of proper intersections (see e.g. [C, §12.3]). □

From this, we can check the counterpart of point i) and ii) of Assumption 10 in the open set M. In § 4.3, we will obtain an open subset of $\mathbb{C}^{N_d^k}$ in the same way.

The next step is to check that we have a dense set of maps where points i), ii) and iv) in Assumption 10 are simultaneously satisfy. We denote by $\chi_{\mathcal{P}(\lambda)}$ (resp. $\chi_{\mathcal{R}(\lambda)}$) the eigenvalue of $D_{\mathcal{P}(\lambda)}\mathcal{F}_\lambda$ (resp. $D_{\mathcal{R}(\lambda)}\mathcal{F}_\lambda^2$) with the smallest modulus. A first observation, already made in Remark 3.15, is that the condition $\overline{\langle \chi_{\mathcal{P}(\lambda)}, \chi_{\mathcal{R}(\lambda)} \rangle} = \mathbb{C}^*$ is equivalent to 1, θ and t being independent over $\mathbb{Q}$, where $\chi_{\mathcal{R}(\lambda)} = e^{2i\pi\theta}\chi_{\mathcal{P}(\lambda)}^t$. This is fulfilled by a dense subsets of $(\theta, t) \in \mathbb{R}^2$. Hence, in order to have i), ii) and iv) in Assumption 10 simultaneously it is sufficient to have the "transversality" property described in Lemma 4.9 below, between two families of hypersurfaces $(Y_{\zeta,t})_{\zeta\in\mathbb{S}^1, t\in\mathbb{R}}$ and $(X_\omega)_{\omega\in\Sigma}$. If $\zeta \in \mathbb{S}^1$ and $t \in \mathbb{R}$, we set

$$Y_{\zeta,t} := \left\{ \lambda \in D(\infty, 1/\delta) \times \hat{M} \ ; \ \chi_{\mathcal{R}(\lambda)} = \zeta \chi_{\mathcal{P}(\lambda)}^t \right\}.$$

Observe that $\chi_{\mathcal{R}(a,\alpha,\beta,\epsilon)} \simeq \alpha^2 - \beta^2$ and $\chi_{\mathcal{P}(a,\alpha,\beta,\epsilon)} \simeq \alpha - \beta$ near $a = \infty$ so $Y_{\zeta,t}$ is actually a hypersurface. The family $(Y_{\zeta,t})_{\zeta\in\mathbb{S}^1, t\in\mathbb{R}}$ defined a (possibly singular) foliation which is not holomorphic. On the other hand, $(X_\omega)_{\omega\in\Sigma}$ is parametrized by a Cantor set and depends continuously on ω. Furthermore, the blender property of $\Lambda(\lambda)$ ensures that $(X_\omega)_{\omega\in\Sigma}$ covers $D(\infty, 1/\delta) \times \hat{M}$. Actually, many points belong to two X_ω and $X_{\omega'}$ at the same time, which will greatly simplify the verification of Assumption 10.

Lemma 4.8. *Possibly by reducing M, for each $\lambda \in M$*

- $W^u_{\mathcal{P}(\lambda),loc} \subset \mathbb{D}_{1/4} \times D(-1, 1/2)$,
- *there exist two words $\omega, \omega' \in \Sigma$ such that $\omega' \neq \omega$ and $\lambda \in X_\omega \cap X_{\omega'}$.*

Proof. The first point simply follows from the facts that $\mathcal{P}(\lambda) \in \mathbb{D}_{1/4} \times D(-1, 1/2)$ and that $W^u_{\mathcal{P}(\lambda),loc}$ is almost a straight vertical graph when $|a|$ is large.

For the second point, observe that both $V_3(a)$ and $V_4(a)$ are contained in $D(-1, 1/2)$ hence, since each H_j in Lemma 4.5 contains $D_{1/4} \subset \overline{\mathbb{D}}$, the local stable manifold $W^u_{\mathcal{P}(\lambda),loc}$ intersects $H_3 \times V_3(a)$ and $H_4 \times V_4(a)$ in two vertical graphs tangent to C_ρ. By Lemma 4.6, there exist two intersections between $W^u_{\mathcal{P}(\lambda),loc}$ and $\Lambda(\lambda)$. □

Lemma 4.9. *Let M be as in Lemma 4.8. Let $\omega, \omega' \in \Sigma$, $\lambda \in M$ and $(\zeta, t) \in \mathbb{S}^1 \times \mathbb{R}$. If there exist irreducible components Z_ω and $Z_{\omega'}$ of X_ω and $X_{\omega'}$ respectively such that $\lambda \in Z_\omega = Z_{\omega'} \subset Y_{\zeta,t}$ then $\omega = \omega'$.*

Proof. In this situation, by Lemma 4.7, Z_ω intersects $\{\infty\} \times \hat{M}$. As we have seen in the proof of Lemma 4.7, a point $\lambda = (\infty, \alpha, \beta, \epsilon) \in \{\infty\} \times \hat{M}$ is in X_ω if and only if

$$\epsilon\mu \left(\frac{\beta}{1-\mu} + (1-\alpha) h_\omega(\mu) \right) = \frac{\epsilon}{\alpha - \beta - 1},$$

where $\mu = (\alpha^2 - \beta^2)^{-1} = \chi_{\mathcal{R}(\lambda)}^{-1}$ and $h_\omega(\mu) = \sum_{n \geq 0} \omega_n \mu^n$. The relation $Z_\omega = Z_{\omega'}$ implies that on $\hat{Z}_\omega := Z_\omega \cap \{\infty\} \times \hat{M}$, which has dimension at least 2, $h_\omega(\mu) = h_{\omega'}(\mu)$. If $\omega \neq \omega'$ then these two power series are different and μ has to be constant on $\hat{Z}_\omega$. On the other hand, $Z_\omega \subset Y_{\zeta,t}$ hence $\chi_{\mathcal{R}(\lambda)} = \zeta \chi_{\mathcal{P}(\lambda)}^t$ on $\hat{Z}_\omega$. Since $\chi_{\mathcal{R}(\infty,\alpha,\beta,\epsilon)} = \alpha^2 - \beta^2 = \mu^{-1}$ and $\chi_{\mathcal{P}(\infty,\alpha,\beta,\epsilon)} = \alpha - \beta$, both are constant on $\hat{Z}_\omega$ and thus α, β are also constant. This contradicts the fact that $\hat{Z}_\omega$ has dimension at least 2. Hence, $\omega = \omega'$. □

4.3. **Higher dimensions and degrees.** The next step is to pass to higher dimensions and degrees. Let $k \geq 2$ and $d \geq 2$. We denote by $[y_0 : \cdots : y_k]$ the homogeneous coordinates on $\mathbb{P}^k$ and we will mainly work on the affine chart $y_0 = 1$. Since the important coordinates for the dynamics are the two first ones, we take the convention of notation that

$$z = y_1, \quad w = y_2 \quad \text{and} \quad y = (y_3, \ldots, y_k).$$

Recall that $N_d^k := (k+1)\binom{k+d}{d}$ is the dimension of the set of $k+1$ homogeneous polynomials of degree d. We choose coordinates in $\mathbb{C}^{N_d^k}$ such that, if $\sigma = (\sigma_3, \ldots, \sigma_k)$ and $\tau = (\tau_3, \ldots, \tau_k)$ are in $\mathbb{C}^{k-2}$ then the parameter $\lambda = (a, \alpha, \beta, \epsilon, \sigma, \tau, 0) \in \mathbb{C}^{N_d^k}$ corresponds to the map

$$f_\lambda(z, w, y) = \left(\alpha z + \epsilon w + \beta z w + w \sum_{i=3}^{k} \tau_i y_i, q_a(w), \sigma_3 y_3, \ldots, \sigma_k y_k \right). \tag{10}$$

Observe that, when $\tau = 0$ then this map is a product map, acting by $\mathcal{F}_{(a,\alpha,\beta,\epsilon)}$ on (z, w) and by a diagonal matrix on y. When $\tau \neq 0$ then it is a skew product of $\mathbb{C} \times \mathbb{C}^{k-1}$. In what follows, we will take σ with $|\sigma_i| > 1$ very large with respect to α, β and ϵ to ensure a dominated splitting. The choice of σ will also depend on a in order to obtain non-resonance conditions for the periodic points p and r. The parameter τ will be chosen very small at the end and its only role will be to obtain the point iii) in Assumption 10.

To be more explicit, let $(a, \alpha, \beta, \epsilon)$ be in the set M given by Lemma 4.7, let $\tau = 0$ and let $\sigma = (\sigma_3, \ldots, \sigma_k) \in \mathbb{C}^{k-2}$ be such that each $|\sigma_i| > 1$ is large. In this situation, the corresponding map f_λ has a fixed saddle point $p(\lambda) = (\mathcal{P}(\lambda), 0)$, a period 2 repelling point $r(\lambda) = (\mathcal{R}(\lambda), 0)$ and a repelling hyperbolic set

$$\Lambda(\lambda) := \bigcap_{n \geq 0} f_\lambda^{-2n}(\mathbb{D}_R \times (V_-(a) \cup V_+(a)) \times \mathbb{D}^{k-2})$$

which is equal to the product of the hyperbolic set of $\mathcal{F}_{(a,\alpha,\beta,\epsilon)}$ with $\{0\}$. If each $|\sigma_i| > 1$ is large enough then the cone field $C_\rho := \{(u_1, \ldots, u_k) \in \mathbb{C}^k \ ; \ \rho|u_1| \leq \max_{2 \leq i \leq k} |u_k|\}$, where ρ is given by Lemma 4.6, is contracted by f_λ (resp. f_λ^2) on $\mathbb{D}_R \times (U_-(a) \cup U_+(a)) \times \mathbb{D}^{k-2}$ (resp. $\mathbb{D}_R \times (V_-(a) \cup V_+(a)) \times \mathbb{D}^{k-2}$) and thus $\Lambda(\lambda)$ has the following blender property: for each $i \in \{1, 2, 3, 4\}$, any vertical graph in $\overline{\mathbb{D}} \times V_i(a) \times \mathbb{D}^{k-2}$ tangent to C_ρ intersects $\Lambda(\lambda)$. Moreover, a simple computation gives that the critical set of f_λ is disjoint from $\mathbb{D}_R \times (U_-(a) \cup U_+(a)) \times \mathbb{D}^{k-2}$ and that the stable manifold of $p(\lambda)$ is equal to $\mathbb{C} \times \{(\tilde{w}(a), 0)\}$, where $\tilde{w}(a)$ is the unique fixed point of q_a in $U_-(a)$.

We also need non-resonance conditions for $p(\lambda)$ and $r(\lambda)$ and for that we will choose $(a, \alpha, \beta, \epsilon) \in M$ and σ more carefully. When $|a|$ is very large then the eigenvalues of $D_{p(\lambda)} f_\lambda$ are close to $\alpha - \beta$, $-2a$ and $\sigma_3, \dots, \sigma_k$. Those of $D_{r(\lambda)} f_\lambda^2$ are close to $\alpha^2 - \beta^2$, $-4a^2$ and $\sigma_3^2, \dots, \sigma_k^2$. In both cases, only the first two ones depend on a. Hence, we can first fix α_1, β_1 and ϵ_1 then $a_1 \in \mathbb{R}_+$ then $\sigma_1 = (\sigma_i)_{3 \le i \le k} \in (\mathbb{R}_-)^{k-2}$ in order to have $(a_1, \alpha_1, \beta_1, \epsilon_1) \in M$ and for f_{λ_1}, $\lambda_1 := (a_1, \alpha_1, \beta_1, \epsilon_1, \sigma_1, 0) \in \mathbb{C}^{N_d^k}$

(1) the eigenvalues of $p(\lambda_1)$ satisfy the *strong Sternberg* condition of order 3,
(2) $\max_{3 \le i \le k} |\sigma_i| < |a_1| < \min_{3 \le i \le k} |\sigma_i|^2/4$,
(3) there is *no resonance* between the eigenvalues of $D_{r(\lambda_1)} f_{\lambda_1}^2$ and they are all different.

A first remark is that we choose a_1 in $\mathbb{R}_+$ and σ_1 in $(\mathbb{R}_-)^{k-2}$ only to have specific cone contractions for Lemma 4.12. For the other properties, recall that there is a resonance between k complex numbers $(\eta_1, \dots, \eta_k)$ if there exist $j \in \{1, \dots, k\}$ and a multi-index $N = (N_1, \dots, N_k)$ of non-negative integers such that $\sum_{i=1}^k N_i \ge 2$ and $\left| \prod_{i=1}^k \eta_i^{N_i} \right| = |\eta_j|$. Notice that for repelling or attracting periodic points, the eigenvalues have no resonance for an open and dense set of parameters, as soon as there is no persistence relations between them, which is our case. In this situation, the periodic point can be holomorphically linearized, with a linearization which depends holomorphically in the parameters. This can be seen in the proof of Lattès in [Lat] for $k = 2$ and Berger and Reinke deal with a much more general setting in [BR]. Observe that we also ask for different eigenvalues in order to locally follow the associated eigenspaces.

In the saddle case, the absence of resonance is no longer an open condition and in particular, it might be not possible to holomorphically linearized in family. Thus, we use the work of Sell [Se] in order to have C^1-linearization in family. The strong Sternberg condition of order 3 comes from [Se] and is implied by the non-existence of resonance with multi-index $N = (N_1, \dots, N_k)$ with $\sum_{i=1}^k N_i \le 3$ which is an open and dense property. The condition **(2)** above ensures that the spectral spreads as defined in [Se] satisfy $\rho^- = 1$ and $\rho^+ < 2$. Hence, [Se, Theorem 7] implies that the dynamics near $p(\lambda)$ can be C^1-linearized for λ in a small neighborhood of λ_1 and with a linearization which depends continuously in the C^1-topology on λ.

Since all the properties above are stable under small C^1-perturbations there exists a small connected open neighborhood $\tilde{M}$ of λ_1 in $\mathbb{C}^{N_d^k}$ such that $p(\lambda)$, $r(\lambda)$ and $\Lambda(\lambda)$ can be followed holomorphically and, for each $\lambda \in \tilde{M}$, in addition to the linearization properties of $p(\lambda)$ and $r(\lambda)$ we just mentioned, we also have the following properties.

- If we set $U_\pm := U_\pm(a_1)$, $V_\pm := V_\pm(a_1)$, $\mathcal{U}_\pm := \mathbb{D}_R \times U_\pm \times \mathbb{D}^{k-2}$ and $\mathcal{V}_\pm := \mathbb{D}_R \times V_\pm \times \mathbb{D}^{k-2}$ then f_λ (resp. f_λ^2) contracts to cone field C_ρ on $\mathcal{U}_+ \cup \mathcal{U}_-$ (resp. on $\mathcal{V}_+ \cup \mathcal{V}_-$) and $\overline{\mathcal{U}_+ \cup \mathcal{U}_-} \subset f_\lambda^2(\mathcal{V}_\pm)$ with f_λ^2 injective and expanding on $\mathcal{V}_+$ and on $\mathcal{V}_-$.
- The critical set of f_λ is disjoint from $\mathcal{U}_+ \cup \mathcal{U}_-$.
- Using inverse branches, each point in $\Lambda(\lambda)$ corresponds to a unique coding $\omega \in \Sigma$ and $(\lambda \mapsto x_\omega(\lambda))_{\omega \in \Sigma}$ gives the holomorphic motion of $\Lambda(\lambda)$.
- For each $i \in \{1, 2, 3, 4\}$, any vertical graph in $\overline{\mathbb{D}} \times V_i(a_1) \times \mathbb{D}^{k-2}$ tangent to C_ρ intersects $\Lambda(\lambda)$.
- $p(\lambda)$ is saddle and $W^u_{p(\lambda),loc}$ is a hypersurface intersecting $\mathbb{D}_{1/4} \times V_- \times \mathbb{D}^{k-2}$ as a vertical graph tangent to C_ρ. In particular, as in Lemma 4.8, $W^u_{p(\lambda),loc}$ intersects $\Lambda(\lambda)$ at two different points.

- The stable manifold $W^s_{p(\lambda)}$ contains a subset close to $\mathbb{D}_R \times \{(\tilde{w}(a_1), 0)\}$ and in particular it intersects any vertical graphs in $\mathcal{U}_-$ tangent to C_ρ.
- $r(\lambda)$ is a repelling 2-periodic point in $\Lambda(\lambda)$ with k different eigenvalues.

In particular, we can define for $\omega \in \Sigma$

$$\tilde{X}_\omega := \left\{ \lambda \in \tilde{M} \ ; \ x_\omega(\lambda) \in W^u_{p(\lambda),loc} \right\},$$

and for $(\zeta, t) \in \mathbb{S}^1 \times \mathbb{R}$, if $\chi_{r(\lambda)}$ (resp. $\chi_{p(\lambda)}$) is the eigenvalue of $D_{r(\lambda)} f^2_\lambda$ (resp. $D_{p(\lambda)} f_\lambda$) with the smallest modulus,

$$\tilde{Y}_{\zeta,t} := \left\{ \lambda \in \tilde{M} \ ; \ \chi_{r(\lambda)} = \zeta \chi^t_{p(\lambda)} \right\}.$$

The following result is easily deduced from its counterpart on M.

Lemma 4.10. *For each $\omega \in \Sigma$ the set $\tilde{X}_\omega$ is a (possible empty) hypersurface of $\tilde{M}$. Moreover, there exists a connected open neighborhood $\tilde{M}' \subset \tilde{M}$ of λ_1 such that if (ζ, t) are in $\mathbb{S}^1 \times \mathbb{R}$ and $\lambda \in \tilde{M}'$ is a regular point of $\tilde{Y}_{\zeta,t}$ then there is $\omega \in \Sigma$ such that λ belongs to an irreducible component of $\tilde{X}_\omega$ which is not included in $\tilde{Y}_{\zeta,t}$.*

Proof. The first point is a direct consequence of Lemma 4.7 since if $\tilde{X}_\omega = \tilde{M}$ then $X_\omega = D(\infty, 1/\delta) \times \hat{M}$.

Exactly as in the proof of Lemma 4.7, there exists a connected open neighborhood λ_1 in $\tilde{M}$ such that if an irreducible component of $\tilde{X}_\omega$ intersects $\tilde{M}'$ then it also intersects $M \times \{(\sigma_1, 0)\}$.

Let $(\zeta, t) \in \mathbb{S}^1 \times \mathbb{R}$ and $\lambda \in \tilde{M}'$ be a regular point of $Y_{\zeta,t}$. As we have already seen, there exists two different coding $\omega, \omega' \in \Sigma$ such that $\lambda \in \tilde{X}_\omega \cap \tilde{X}_{\omega'}$. Let $\tilde{Z}_\omega$ and $\tilde{Z}_{\omega'}$ be irreducible components of $\tilde{X}_\omega$ and $\tilde{X}_{\omega'}$ respectively containing λ. Assume by contradiction that both $\tilde{X}_\omega$ and $\tilde{X}_{\omega'}$ are contained in $\tilde{Y}_{\zeta,t}$. As λ is a regular point of $\tilde{Y}_{\zeta,t}$ this implies that $\tilde{Z}_\omega = \tilde{Z}_{\omega'} \subset \tilde{Y}_{\zeta,t}$. Hence, since $\tilde{Z}_\omega$ intersects $M \times \{(\sigma_1, 0)\}$, a similar result holds on M which is not possible by Lemma 4.9. □

As a consequence, generically $p(\lambda)$ has plenty of homoclinic points.

Lemma 4.11. *The set of parameters in $\tilde{M}$ where Assumption 8 holds is open and dense.*

Proof. This set is clearly open. It remains to prove that it is dense. Notice that the set Σ' of $\omega \in \Sigma$ coding for a point with dense orbit in $\Lambda(\lambda)$ is dense in Σ and does not depend on λ. As each $\tilde{X}_\omega$ is a hypersurface, the set $\cup_{\omega \in \Sigma'} \tilde{X}_\omega$ is dense in $\tilde{M}$. Let $\omega \in \Sigma'$, $\lambda \in \tilde{X}_\omega$ and let Γ be a small neighborhood of $x_\omega(\lambda)$ in $W^u_{p(\lambda),loc}$. For $n \geq 1$ large enough, its image $f^{2n}_\lambda(\Gamma)$ contains a vertical graph in $\mathcal{U}_-$ and thus, as we have seen when $\tilde{M}$ was chosen, it intersects $W^s_{p(\lambda)}$. As the graph is vertical, the intersection is transverse. Moreover, we obtain in this way several different intersection points. Actually, the orbit of $x_{\omega(\lambda)}$ is dense in $\Lambda(\lambda)$ and, by the blender property, the projection on the first coordinate of this set contains $\overline{\mathbb{D}}$. This, combined to the fact that the graphs above are tangent to C_ρ with $\rho > 100$, ensures that several of these intersection points are different from $p(\lambda)$.

Finally, all the dynamics above stay in $\mathcal{U}_-$ which is disjoint from the critical set of f_λ, by assumption on $\tilde{M}$. □

4.4. **Tangencial dynamics.** In this part, we will prove that a property, which is robust in the C^1-topology and which implies iii) in Assumption 10, holds generically in the open set $\tilde{M} \subset \mathbb{C}^{N_d^k}$ obtained in § 4.3.

To fix some notations, let $\lambda \in \tilde{M}$ and $g_\lambda := f_\lambda^2$. Since $\Lambda(\lambda)$ is a hyperbolic set for g_λ with a dominated splitting, to each history $\hat{x}(\lambda) = (x_n)_{n\leq 0}$ in the natural extension $\hat{\Lambda}(\lambda)$ is associated a strong unstable subspace $E^{uu}_{\hat{x}(\lambda)}$. This subspace is simply obtained by

$$E^{uu}_{\hat{x}(\lambda)} := \lim_{n\to\infty} D_{x_{-n}} g_\lambda^n E^v = \lim_{n\to\infty} (D_{x_{-1}} g_\lambda \circ \cdots \circ D_{x_{-n}} g_\lambda) E^v. \tag{11}$$

where $E^v = \{(u_1, \ldots, u_k) \in \mathbb{C}^k \ ; \ u_1 = 0\}$. The strong unstable manifold can be constructed in a similar way using graph transform but we will not use it. These objects depend continuously on $\hat{x}$ and holomorphically on $\lambda \in \tilde{M}$. Actually, this is true as long as the hyperbolic set can be followed, a remark that will be used in Lemma 4.12.

Observe that the natural extension $\hat{\Lambda}(\lambda)$ of $\Lambda(\lambda)$ corresponds to the two-sided full shift encoded by $\hat{\Sigma} := \{-1, 1\}^{\mathbb{Z}}$. For $l \in \mathbb{N}$ and $n \in \mathbb{Z}$, we set $\omega_n(l) = 1$ if $n < l$ and $\omega_n(l) = -1$ otherwise. If $\omega(l) := (\omega_n(l))_{n\in\mathbb{N}}$ and $\hat{\omega}(l) := (\omega_n(l))_{n\in\mathbb{Z}}$ then $x_{\omega(l)}(\lambda) \in \Lambda(\lambda)$ is a preimage of $r(\lambda)$ by g_λ^l and $x_{\hat{\omega}(l)}(\lambda) \in \hat{\Lambda}(\lambda)$ is a history of $x_{\omega(l)}(\lambda)$.

Lemma 4.12. *The set T defined by*

$$T := \left\{ \lambda \in \tilde{M} \ ; \ E^{uu}_{x_{\hat{\omega}(0)}(\lambda)} \text{ contains an eigenvector of } D_{r(\lambda)} g_\lambda \right\}$$

is a proper analytic subset of $\tilde{M}$.

Proof. First, recall that the eigenvalues of $D_{r(\lambda)} g_\lambda$ are all different thus we can follow the corresponding eigenspaces holomorphically in λ. Hence, the fact that T is analytic simply comes from the holomorphic dependency of $E^{uu}_{\hat{x}(\lambda)}$ on λ. It remains to prove properness, i.e. to find $\lambda \in \tilde{M}$ outside T. Observe that as soon as $k \geq 3$, the parameter $\lambda_1 = (a_1, \alpha_1, \beta_1, \epsilon_1, \sigma_1, 0) \in \mathbb{C}^{N_d^k}$ defined in § 4.3 is in T since f_{λ_1} is a product map on $\mathbb{C}^2 \times \mathbb{C}^{k-2}$. We perturb it as a skew product of $\mathbb{C} \times \mathbb{C}^{k-1}$

$$f_{\lambda_2}(z, w, y) = \left(\alpha_1 z + \epsilon_1 w + \beta_1 zw + w \sum_{i=3}^{k} \tau_i y_i, q_{a_1}(w), \sigma_3 y_3, \ldots, \sigma_k y_k \right),$$

where $\lambda_2 := (a_1, \alpha_1, \beta_1, \epsilon_1, \sigma_1, \tau_1, 0) \in \mathbb{C}^{N_d^k}$, with $\tau_1 = (\tau_i)_{3\leq i\leq k} \in (\mathbb{R}_{>0})^{k-2}$ is small enough to have $\lambda_2 \in \tilde{M}$. We will deform f_{λ_2} along a path $(f_{\gamma(t)})_{t\in[0,1]}$ of maps of the form (10) such that $f_{\gamma(1)} = f_{\lambda_2}$ and $f_{\gamma(0)}$ does not satisfies the property defining T. Observe that the path γ may go outside $\tilde{M}$ but the hyperbolic set Λ can be followed all along γ, which is sufficient.

For $t \in [0, 1]$ we define $\gamma(t) := (a_1, |\alpha_1| e^{it \arg(\alpha_1)}, t\beta_1, |\epsilon_1| e^{it \arg(\epsilon_1)}, \sigma_1, \tau_1, 0)$. This path is chosen in such a way that $\gamma(0) = (a_1, |\alpha_1|, 0, |\epsilon_1|, \sigma_1, \tau_1, 0)$ is in $(\mathbb{R}_+)^{N_d^k}$ and that the map $g_{\gamma(t)}$ is still expanding on $\mathcal{V}_+ \cup \mathcal{V}_-$ with $\overline{\mathcal{V}_+ \cup \mathcal{V}_-} \subset g_{\gamma(t)}(\mathcal{V}_+) \cap g_{\gamma(t)}(\mathcal{V}_-)$ since $(\alpha_1, \beta_1, \epsilon_1)$ satisfies (7). In particular, the hyperbolic set Λ can be followed in a neighborhood of this path.

From this, an important remark is that, by Lemma 4.4, $r(\gamma(0)) = (z_0, w_0, 0)$ with w_0 real close to -1 and, for $l \geq 1$, $x_{\omega(l)}(\gamma(0)) = (z_l, w_l, 0)$ with w_l real close to 1. Moreover, if $(z, w) \in \mathbb{D}_R \times (V_- \cup V_+)$ with w real then $D_{(z,w,0)} g_{\gamma(0)} = (A_{i,j})$ is a real matrix with $A_{i,j} = 0$ if $i \neq j$ and $i \neq 1$, and, with the notation $w' := q_{a_1}(w)$,

$A_{1,1} = |\alpha_1|^2$, $A_{2,2} = 4a_1^2 ww'$, $A_{1,2} = |\epsilon_1|(|\alpha_1| + 2a_1 w)$, $A_{i,i} = \sigma_i^2$ and $A_{1,i} = \tau_i(|\alpha_1| w + \sigma_i w')$

for $i \in \{3, \dots, k\}$. From this, using that $r(\gamma(0)) = (z_0, w_0, 0)$ with w_0 close to -1 and w_0' close to 1, it is easy to see that the eigenvectors associated to $r(\gamma(0))$ are proportional to e_1 and to $e_i + b_i e_1$ where $b_i > 0$ for each $i \in \{2, \dots, k\}$. Here, $(e_1, \dots, e_k)$ is the canonical basis of $\mathbb{C}^k$. On the other hand, for each $l \geq 1$, $x_{\omega(l)}(\gamma(0)) = (z_l, w_l, 0)$ with w_l real close to 1 and w_l' close to -1. Hence, a vector of the form $e_i + c_i e_1$ with $-1 \leq c_i \leq 0$ is send by $D_{(z_l, w_l, 0)} g_{\gamma(0)}$ on a vector proportional to $e_i + c_i' e_1$ with

$$c_i' = \frac{|\alpha_1|^2 c_i + \tau_i(|\alpha_1| w_l + \sigma_i w_l')}{\sigma_i^2} \text{ if } i \in \{3, \dots, k\} \text{ and } c_2' = \frac{|\alpha_1|^2 c_2 + |\epsilon_1|(|\alpha_1| + 2a_1 w_l)}{4a_1^2 w_l w_l'}.$$

In particular, since a_1 and every σ_i are very large and $|\alpha_1| < 2$, $|\epsilon_1| > 1/20$, each c_i' satisfies $-1 \leq c_i' < 0$. This implies that the hyperspace $E^{uu}_{x_{\hat{\omega}(0)}(\gamma(0))}$, which by (11) is equal to

$$\lim_{n \to \infty} \left(D_{x_{\omega(1)}(\gamma(0))} g_{\gamma(0)} \circ \cdots \circ D_{x_{\omega(n)}(\gamma(0))} g_{\gamma(0)} \right) E^v,$$

is generated by $(e_i + d_i e_1)_{2 \leq i \leq k}$ with $d_i < 0$. Hence, it contains none of the eigenvectors of $D_{r(\gamma(0))} g_{\gamma(0)}$ described above. This conclude the proof. □

4.5. **Verification of the assumptions.** We have now all the ingredients to prove Theorem 4.1. Let λ_1, $\tilde{M}$, $U_\pm$, $\mathcal{U}_\pm$, $V_\pm$ and $\mathcal{V}_\pm$ be as in § 4.3. Recall that f_{λ_1} is of the form

$$f_{\lambda_1}(z, w, y) = \left(\alpha_1 z + \epsilon_1 w + \beta_1 z w + w \sum_{i=3}^{k} \tau_i y_i, q_a(w), \sigma_3 y_3, \dots, \sigma_k y_k \right).$$

As we have said in the discussion in § 4.3 where $\tilde{M}$ was chosen, for each $\lambda \in \tilde{M}$ the map f_λ satisfies several assumptions of § 3.2. More precisely, if $\rho > 0$ and $R > 0$ as in Lemma 4.6, then we already know that Assumptions 1, 3 and 4 hold in $\tilde{M}$. This is also true for Assumption 2 except on the point about the small Julia set J_k, which is not necessarily well-defined for f_λ, and for Assumption 6 except the part about the domain of linearization. For this last point, since the map f_λ^2 restricted to $\mathcal{V}_-$ is expanding and injective with $\mathcal{U}_-$ in its image so, if $h_\lambda \colon \mathcal{U}_- \to \mathcal{V}_-$ denotes its inverse then we can extend the linearization δ_λ on $\mathcal{U}_-$ in an injective way using the dynamics, i.e. if $x \in \mathcal{U}_-$ then $\delta_\lambda(x) := (D_{r(\lambda)} f_\lambda^2)^n \circ \delta_\lambda \circ h_\lambda^n(x)$ for $n \geq 1$ large enough. We will come back to the small Julia set of Assumption 2 latter.

Assumption 5 is just a reformulation of the fact that $\tilde{X}_\omega$ is a proper analytic subset of $\tilde{M}$ and thus, it holds on $\tilde{M}$ by Lemma 4.10.

For the other assumptions, we consider a perturbation $f_{\tilde{\lambda}}$ in $\tilde{M}'$ of f_{λ_1} defined by

$$f_{\tilde{\lambda}}(z, w, y) = f_{\lambda_1}(z, w, y) + c(z^d, w^d, y_3^d, \dots, y_k^d),$$

where $c \in \mathbb{C}^*$ is very close to 0. Observe that $f_{\tilde{\lambda}}$ is a regular polynomial endomorphism of $\mathbb{C}^k$ which is a skew product of $\mathbb{C} \times \mathbb{C}^{k-1}$ above a product map of $\mathbb{C}^{k-1}$. This implies that the small Julia set $J_k(f_{\tilde{\lambda}})$ is exactly the closure of the repelling periodic points of $f_{\tilde{\lambda}}$. Actually, this was proved by Jonsson [J] when $k = 2$ where the key ingredient is a fibered formula for the equilibrium measure. This formula has been generalized in higher dimension by [DT, Corollary 1.2] and thus the result of Jonsson also holds for $f_{\tilde{\lambda}}$. In particular, since the repelling periodic points are dense in the hyperbolic set $\Lambda(\tilde{\lambda})$, this gives that $\Lambda(\tilde{\lambda}) \subset J_k(f_{\tilde{\lambda}})$. As this property persists under small perturbations (see e.g. [Du, Lemma 2.3]), Assumption 2 holds in a small neighborhood of $f_{\tilde{\lambda}}$ in End_d^k. Moreover,

one part of the critical set of $f_{\tilde\lambda}$ comes from the dynamics on the basis $\mathbb{C}^{k-1}$. A simple computation gives that the remaining part of this critical set is

$$C_{\tilde\lambda} := \left\{ (z,w,y) \in \mathbb{C}^k \ ; \ w = -\frac{\alpha_1 + cdz^{d-1}}{\beta_1} \right\},$$

which is always transverse to fibers of the form $\{w = w_0,\ y = y_0\}$ except when $d \geq 3$ and $w_0 = -\alpha_1/\beta_1$. The stable manifold $W^s_{p(\tilde\lambda)}$ of the saddle point $p(\tilde\lambda)$ is an attracting basin in the invariant fiber $\{w = w_0, y = 0\}$ where $w_0 \simeq -1$ is the unique fixed point of $w \mapsto q_{a_1}(w) + cw^d$ in U_- and thus by a classical result of Fatou it has to intersect $C_{\tilde\lambda}$ in a point of infinite orbit. Furthermore, since $w_0 \simeq -1$ and $-\alpha_1/\beta_1$ has large modulus, they cannot be equal. This gives a transverse intersection between $W^s_{p(\tilde\lambda)}$ and the critical set. The skew product structure of $f_{\tilde\lambda}$ and the fact that $\{w = w_0, y = 0\}$ is not a critical fiber ensure that the images of this transverse intersection stay transversal. This shows Assumptions 7 is satisfied in a small neighborhood of $f_{\tilde\lambda}$.

Observe that by Lemma 4.11 and Lemma 4.12, there exists near $\tilde\lambda$ a small non-empty open set Ω in $\mathrm{End}^k_d \cap \tilde M'$ where, in addition to all the assumptions above, Assumption 8 holds and which is disjoint from the set T defined in Lemma 4.12. Moreover, as by [FJ, Corollary C] when $k = 2$ and [DS3, Lemma 5.4.5] for $k \geq 2$ the exceptional set is generically empty (or reduced to the hyperplane at infinity in the case of regular polynomial endomorphisms of $\mathbb{C}^k$, with the same proof than [DS3, Lemma 5.4.5]), we can also assume that it is the case for maps in Ω, i.e. that Assumption 9 holds on Ω.

Hence, all assumptions of § 3.2 are satisfied on Ω, except possibly Assumption 10. Let Ω' a non-empty open subset of Ω. Let $\lambda' \in \Omega'$ be a regular point of the foliation $(\tilde Y_{\zeta,t})_{(\zeta,t)\in\mathbb{S}^1\times\mathbb{R}}$ defined just before Lemma 4.10. This lemma implies that there exists $\omega' = (\omega'_n)_{n\geq 0} \in \Sigma$ such that $\lambda' \in \tilde X_{\omega'}$ and $\tilde X_{\omega'}$ is not contained in some $\tilde Y_{\zeta,t}$. In particular, if $\omega = (\omega_n)_{n\geq 0}$ is very close to ω' then $\tilde X_\omega$ intersects Ω' and is not contained in some $\tilde Y_{\zeta,t}$. We defined such ω by

$$\omega_n = \omega'_n \text{ if } n \leq N_1, \quad \omega_n = 1 \text{ if } N_1 < n \leq N_1 + N_2 \text{ and } \omega_n = -1 \text{ if } n > N_1 + N_2,$$

where N_1 and N_2 are two very large positive integers. The first condition ensures that ω is close enough to ω'. The third one that the corresponding point $x_\omega(\lambda) \in \Lambda(\lambda)$ is a preimage by $f_\lambda^{2(N_1+N_2+1)}$ of the repelling periodic point $r(\lambda)$. This gives point i) of Assumption 10 with $m := 2(N_1 + N_2 + 1)$ for every $\lambda \in \Omega' \cap \tilde X_\omega$. By definition of $\tilde X_\omega$, $x_\omega(\lambda) \in W^u_{p(\lambda),loc}$ on this set and thus ii) also holds. The fact that $\tilde X_\omega$ is not contained in some $\tilde Y_{\zeta,t}$ implies that iv) is satisfied on a dense subset of $\Omega' \cap \tilde X_\omega$. Finally, recall Ω' is disjoint from the set T defined in Lemma 4.12. Recall that this implies that if $\omega(1) := (\omega_n(1))_{n\geq 0} \in \Sigma$ and the history $\hat\omega(1) := (\omega_n(1))_{n\in\mathbb{Z}} \in \{-1,1\}^{\mathbb{Z}}$ are defined by $\omega_n(1) = 1$ if $n < 1$ and $\omega_n(1) = -1$ otherwise then the image by $D_{x_{\omega(1)}(\lambda)} f^2_\lambda$ of the strong unstable subspace $E^{uu}_{x_{\hat\omega(1)}(\lambda)}$ associated to $x_{\hat\omega(1)}(\lambda)$ is a generic hyperplane for $D_{r(\lambda)} f^2_\lambda$, i.e. does not contain any eigenvector of $D_{r(\lambda)} f^2_\lambda$. On the other hand, if $\lambda \in \Omega' \cap \tilde X_\omega$ and if E denotes the tangent space of $W^u_{p(\lambda),loc}$ at $x_\omega(\lambda)$ then E is in the cone C_ρ and $D_{x_\omega(\lambda)} f_\lambda^{2(N_1+N_2)} E$ is very close to $E^{uu}_{x_{\hat\omega(1)}(\lambda)}$ if N_2 is large enough. Hence, $D_{x_\omega(\lambda)} f_\lambda^{2(N_1+N_2+1)} E$ is also a generic hyperplane for $D_{r(\lambda)} f^2_\lambda$. Thus, the point iii) of Assumption 10 is satisfied for all $\lambda \in \Omega' \cap \tilde X_\omega$.

In conclusion, the open set Ω verifies all the assumptions of § 3.2.

5. The fundamental height inequalities

5.1. **Adelic metrics, height functions.** Let X be a projective variety, and let $L_0, \dots, L_k$ be $\mathbb{Q}$-line bundle on X, all defined over a number field $\mathbb{K}$.

For any i, assume L_i is equipped with an adelic continuous metric $\{\|\cdot\|_{v,i}\}_{v\in M_{\mathbb{K}}}$ and we denote $\bar{L}_i := (L_i, \{\|\cdot\|_v\}_{v\in M_{\mathbb{K}}})$. Assume also $\bar{L}_i$ is semi-positive. Fix a place $v \in M_{\mathbb{K}}$. Denote by X_v^{an} the Berkovich analytification of X at the place v. We also let $c_1(\bar{L}_i)_v$ be the curvature form of the metric $\|\cdot\|_{v,i}$ on $L_{i,v}^{\mathrm{an}}$.

For any closed subvariety Y of dimension q, as observed by Chambert-Loir [CL], the arithmetic intersection number $\big(\bar{L}_0 \cdots \bar{L}_q | Y\big)$ is symmetric and multilinear with respect to the $\bar{L}_i$'s and is defined inductively by

$$\big(\bar{L}_0 \cdots \bar{L}_q | Y\big) = \big(\bar{L}_1 \cdots \bar{L}_q | \mathrm{div}(s) \cap Y\big) + \sum_{v\in M_{\mathbb{K}}} n_v \int_{Y_v^{\mathrm{an}}} \log \|s\|_{v,0}^{-1} \bigwedge_{j=1}^{q} c_1(\bar{L}_i)_v,$$

for any global section $s \in H^0(X, L_0)$. In particular, if L_0 is the trivial bundle and $\|\cdot\|_{v,0}$ is the trivial metric at all places but v_0, this gives

$$\big(\bar{L}_0 \cdots \bar{L}_k | Y\big) = n_{v_0} \int_{Y_{v_0}^{\mathrm{an}}} \log \|s\|_{v_0,0}^{-1} \bigwedge_{j=1}^{q} c_1(\bar{L}_i)_{v_0}.$$

When L is a big and nef $\mathbb{Q}$-line bundle endowed with a semi-positive continuous adelic metric $\bar{L}$, following Zhang [Zha1], we can define $h_{\bar{L}}(Y)$ as

$$h_{\bar{L}}(Y) := \frac{\big(\bar{L}^{q+1} | Y\big)}{(q+1)[\mathbb{K} : \mathbb{Q}] \deg_Y(L)},$$

where $\deg_Y(L) = (L|_Y)^k$ is the volume of the line bundle L restricted to Y.

Recall that a sequence $(x_i)_i$ of points of $Y(\bar{\mathbb{Q}})$ is *generic* if for any closed subvariety $W \subset Y$ defined over $\mathbb{K}$, there is $i_0 \geq 1$ such that $\mathsf{O}(x_i) \cap W = \varnothing$ for all $i \geq i_0$. By Zhang's inequalities [Zha1], if $h_{\bar{L}} \geq 0$ on $X(\bar{\mathbb{Q}})$, if we let

$$e_1(\bar{L}) := \sup_{Z \subsetneq Y} \inf_{x \in (Y\setminus Z)(\bar{\mathbb{Q}})} h_{\bar{L}}(x),$$

where Z ranges on strict subvarieties of Y defined over $\bar{\mathbb{Q}}$, then we have

$$e_1(\bar{L}) \geq h_{\bar{L}}(Y) \geq \frac{1}{q+1} e_1(\bar{L}). \tag{12}$$

In particular, there is a generic sequence $(x_i)_i$ of closed points of $Y(\bar{\mathbb{Q}})$ such that

$$\liminf_{i\to\infty} h_{\bar{L}}(x_i) \geq h_{\bar{L}}(Y) \geq \frac{1}{q+1} \liminf_{j\to\infty} h_{\bar{L}}(x_j). \tag{13}$$

Let X be a projective variety defined on a number field $\mathbb{K}$ and let $\bar{L}$ be an ample line bundle on X endowed with an adelic semi-positive metric. Let $m \geq 1$ be an integer and, for $1 \leq i \leq m$, let $p_i : X^m \to X$ be the projection onto the i-th factor. Let $\bar{L}_m := p_1^*(\bar{L}) + \cdots + p_m^*(\bar{L})$.

We will use the next lemma.

Lemma 5.1. *For any subvariety $Y \subset X$ defined over $\mathbb{K}$, we have*

$$h_{\bar{L}_m}(Y^m) = m \cdot h_{\bar{L}}(Y).$$

Proof. For $m = 1$, or if $q = 0$ there is nothing to prove. We can assume that L is very ample. Fix $m \geq 2$ and set $q := \dim Y \geq 1$. For any $1 \leq i \leq m$ and any line bundles $M_{q+2}, \dots, M_{qm}$ on X^m, we have

$$\big(c_1(p_i^* L)^{q+1} \cdot c_1(M_{q+2}) \cdots c_1(M_{qm}) \cdot \{Y^m\}\big) = 0.$$

In particular, $\deg_{L_m}(Y^m) = \binom{qm}{q} \Big(c_1(p_1^* L)^q \cdots (c_1(p_2^* L) + \cdots + c_1(p_m^* L))^{q(m-1)} \cdot \{Y^m\}\Big)$, whence

$$\deg_{L_m}(Y^m) = \binom{qm}{q} \deg_L(Y) \deg_{L_{m-1}}(Y^{m-1}).$$

Similarly, as the arithmetic intersection product is multilinear and symmetric, if we let $\pi_i : X^m \to X^{m-1}$ be the cancellation of the ith variable, we have

$$(\bar{L}_m^{qm+1} | Y^m) = \sum_{i=1}^{m} \binom{qm+1}{q+1} (p_i^* \bar{L}^{q+1} \cdot \pi_i^* \bar{L}_{m-1}^{q(m-1)} | Y^m).$$

Let $s_1, \dots, s_{q+1}$ be sections of L such that $\operatorname{div}(s_1) \cap \cdots \cap \operatorname{div}(s_{q+1}) \cap Y = \varnothing$ and let $Z_0 := Y$ and for $1 \leq j \leq q$, $Z_j := Z_{j-1} \cap \operatorname{div}(s_j)$. Following [], as $Y^m = p_i^{-1}(Y) \cap \pi_i^{-1}(Y^{m-1})$, we have

$$\begin{aligned}(p_i^* \bar{L}^{q+1} \cdot \pi_i^* \bar{L}_{m-1}^{q(m-1)} | Y^m) = \quad & (p_i^* \bar{L}^q \cdot \pi_i^* \bar{L}_{m-1}^{q(m-1)} | \pi_i^{-1}(Y^{m-1}) \cap p_i^{-1}(Z_1)) \\ & + \sum_{v \in M_{\mathbb{K}}} n_v \int_{(Y^m)_v^{\mathrm{an}}} \log \|p_i^* s_1\|_{p_i^* \bar{L}, v} c_1(p_i^* \bar{L})_v^q \wedge c_1(\pi_i^* \bar{L}_m)_v^{q(m-1)}\end{aligned}$$

which rewrites as

$$\begin{aligned}(p_i^* \bar{L}^{q+1} \cdot \pi_i^* \bar{L}_{m-1}^{q(m-1)} | Y^m) = \quad & (p_i^* \bar{L}^q \cdot \pi_i^* \bar{L}_{m-1}^{q(m-1)} | \pi_i^{-1}(Y^{m-1}) \cap p_i^{-1}(Z_1)) \\ & + \deg_{L_{m-1}}(Y^{m-1}) \sum_{v \in M_{\mathbb{K}}} n_v \int_{Y_v^{\mathrm{an}}} \log \|s_1\|_{\bar{L}, v} c_1(\bar{L})_v^q.\end{aligned}$$

Similarly, for any $1 \leq j \leq q-1$ one can write

$$\begin{aligned}(p_i^* \bar{L}^{q-j+2} \cdot \pi_i^* \bar{L}_{m-1}^{q(m-1)} | \pi_i^{-1}(Y^{m-1}) \cap p_i^{-1}(Z_{j-1})) = & \\ (p_i^* \bar{L}^{q-j+1} \cdot \pi_i^* \bar{L}_{m-1}^{q(m-1)} | \pi_i^{-1}(Y^{m-1}) \cap p_i^{-1}(Z_j)) & \\ + \deg_{L_{m-1}}(Y^{m-1}) \sum_{v \in M_{\mathbb{K}}} n_v \int_{(Z_{j-1})_v^{\mathrm{an}}} \log \|s_j\|_{\bar{L}, v} c_1(\bar{L})_v^{q-j+1}. &\end{aligned}$$

Summing up over the $q+1$ terms we get

$$(p_i^* \bar{L}^{q+1} \cdot \pi_i^* \bar{L}_{m-1}^{q(m-1)} | Y^m) = \deg_{L_{m-1}}(Y^{m-1}) \cdot (\bar{L}^{q+1} | Y).$$

Together with the above, this gives

$$(\bar{L}_m^{qm+1} | Y^m) = m \tbinom{qm+1}{q+1} \deg_{L_{m-1}}(Y^{m-1}) \cdot (\bar{L}^{q+1} | Y).$$

Since by definition,

$$h_{\bar{L}}(Y) = \frac{(\bar{L}^{q+1} | Y)}{[\mathbb{K} : \mathbb{Q}](q+1) \deg_L(Y)} \quad \text{and} \quad h_{\bar{L}_m}(Y^m) = \frac{(\bar{L}_m^{qm+1} | Y^m)}{[\mathbb{K} : \mathbb{Q}](qm+1) \deg_{L_m}(Y^m)},$$

the proof is complete. □

5.2. **Dynamics over number fields.** Let X be a projective variety, $f : X \to X$ a morphism and L be an ample line bundle on X, all defined over a number field $\mathbb{K}$. Recall that we say (X, f, L) is a *polarized endomorphism* of degree $d > 1$ if $f^*L \simeq L^{\otimes d}$, i.e. f^*L is linearly equivalent to $L^{\otimes d}$. Let $k := \dim X$.

It is known that polarized endomorphisms defined over the field $\mathbb{K}$ admit a *canonical metric*. This is an adelic semi-positive continuous metric on L, which can be built as follows: let $\mathscr{X} \to \mathrm{Spec}(\mathscr{O}_{\mathbb{K}})$ be an $\mathscr{O}_{\mathbb{K}}$-model of X and $\bar{\mathscr{L}}$ be a model of L endowed with a model metric, for example $\bar{\mathscr{L}} = \iota^*\bar{\mathcal{O}}_{\mathbb{P}^N}(1)$, where $\iota : X \hookrightarrow \mathbb{P}^N$ is an embedding inducing L and $\bar{\mathcal{O}}_{\mathbb{P}^N}(1)$ is the naive metrization. We then can define $\bar{L}$ as

$$\bar{L} := \lim_{n\to\infty} \frac{1}{d^n}(f^n)^*\bar{\mathscr{L}}|_{\mathbb{K}}.$$

This metrization induces the *canonical height* $\widehat{h}_f$ of f: for any closed point $x \in X(\bar{\mathbb{Q}})$ and any section $\sigma \in H^0(X, L)$ which does not vanish at x, we let

$$\widehat{h}_f(x) := \frac{1}{[\mathbb{K} : \mathbb{Q}]\deg(x)} \sum_{v\in M_{\mathbb{K}}} \sum_{y\in \mathsf{O}(x)} n_v \log \|\sigma(y)\|_v^{-1},$$

where $\mathsf{O}(x)$ is the Galois orbit of x in X. The function $\widehat{h}_f : X(\bar{\mathbb{Q}}) \to \mathbb{R}$ satisfies $\widehat{h}_f \circ f = d \cdot \widehat{h}_f$, $\widehat{h}_f \geq 0$ and $\widehat{h}_f(x) = 0$ if and only if x is preperiodic under iteration of f, i.e. if there are $n > m \geq 0$ such that $f^n(x) = f^m(x)$. Note that $\widehat{h}_f$ can also be defined as

$$\widehat{h}_f(x) = \lim_{n\to\infty} \frac{1}{d^n} h_{X,L}(f^n(x)),$$

where $h_{X,L}$ is any Weil height function on X associated with the ample line bundle L.

If Y is a subvariety of dimension $q \geq 0$ defined over $\bar{\mathbb{Q}}$, we define

$$\widehat{h}_f(Y) := h_{\bar{L}}(Y) = \frac{\left(\bar{L}^{q+1}|Y\right)}{(q+1)[\mathbb{K} : \mathbb{Q}]\deg_Y(L)}$$

(observe that when $Y = \{x\}$ has dimension 0, both definitions coincide i.e. both definitions of $\widehat{h}_f$ coincide). This satisfies $\widehat{h}_f(f_*(Y)) = d\widehat{h}_f(Y)$, where $f_*(Y)$ is the image of Y by f counted with multiplicity as a cycle on X. In particular, if Y is preperiodic under iteration of f, i.e. if there are $n > m \geq 0$ such that $f^n(Y) = f^m(Y)$, then $\widehat{h}_f(Y) = 0$.

5.3. **Canonical height and height on the base.** We now let $(\mathcal{X}, f, \mathcal{L}, \mathcal{Y})$ be a dynamical pair of degree $d \geq 2$ parametrized by a smooth projective variety S, with regular part $S^0_{\mathcal{Y}}$. Let $\mathcal{Y}^0 := \pi|_{\mathcal{Y}}^{-1}(S^0_{\mathcal{Y}})$. We also assume $(\mathcal{X}, f, \mathcal{L})$, $\mathcal{Y}$ and S are all defined over a number field $\mathbb{K}$. In what follow, we fix an embedding $\iota : \mathbb{K} \hookrightarrow \mathbb{C}$ for which we define the different bifurcation currents.

Definition 5.2. *Let $m \geq \dim S$. If the measure $\mu_{f,\mathcal{Y}}$ is non-zero, we define the m-*higher order canonical height $\widehat{\mathcal{H}}^{(m)}_{f,\mathcal{M}}(\mathcal{Y})$ *of the family $\mathcal{Y}$, relative to $\mathcal{M}$, as*

$$\widehat{\mathcal{H}}^{(m)}_{f,\mathcal{M}}(\mathcal{Y}) := \frac{\mathrm{Vol}^{(m)}_f(\mathcal{Y})}{\dim \mathcal{Y}^{[m]} \cdot \deg^{(m)}_{f,\mathcal{M}}(\mathcal{Y})}.$$

Otherwise, we let $\widehat{\mathcal{H}}^{(m)}_{f,\mathcal{M}}(\mathcal{Y}) := 0$.

Remark 5.3. (1) Observe that both $\mathrm{Vol}_f^{(m)}(\mathcal{Y})$ and $\deg_{f,\mathcal{M}}^{(m)}(\mathcal{Y})$ are geometric quantities that do not depend on the choice of a place (hence we can take another embedding $\iota : \mathbb{K} \hookrightarrow \mathbb{C}$).

(2) If $\dim S = 1$, we have $\widehat{\mathcal{H}}_f^{(1)}(\mathcal{Y}) = \widehat{h}_{f_\eta}(Y_\eta)$,

(3) The quantity $\widehat{\mathcal{H}}_{f,\mathcal{M}}^{(m)}(\mathcal{Y})$ is well-defined by Proposition 1.7 and satisfies $\widehat{\mathcal{H}}_{f,\mathcal{M}}^{(m)}(\mathcal{Y}) > 0$ for all $m \geq \dim S$ if and only if $\mu_{f,\mathcal{Y}}$ is non-zero.

We prove here the following which is inspired from [GH, Theorem 1.4 and Proposition 10.1] and [DGH, Theorem 1.6]:

Theorem 5.4. *Assume that $\mu_{f,\mathcal{Y}}$ is non-zero. Let $m \geq \dim S$ and $\mathcal{M}$ be any ample $\mathbb{Q}$-line bundle on S of volume 1. Then, for any $0 < \varepsilon < \widehat{\mathcal{H}}_{f,\mathcal{M}}^{(m)}(\mathcal{Y})$, there is a non-empty Zariski open subset $\mathcal{U} \subset (\mathcal{Y}^{[m]})^0$ and a constant $C \geq 1$ depending only on $(\mathcal{X}, f, \mathcal{L})$, $\mathcal{Y}$, $\mathcal{M}$, m and ε such that*

$$h_{S,\mathcal{M}}(\pi_{[m]}(x)) \leq \frac{1}{\widehat{\mathcal{H}}_{f,\mathcal{M}}^{(m)}(\mathcal{Y}) - \varepsilon} \sum_{j=1}^{m} \widehat{h}_{f_{\pi(x_j)}}(x_j) + C,$$

for any $x = (x_1, \ldots, x_m) \in \mathcal{U}(\bar{\mathbb{Q}})$.

We need here to mention that one can use the theory of adelic line bundles on quasi-projective varieties set up by Yuan and Zhang [YZ] to obtain such an inequality (see Theorem 6.2.2 therein). However, the proof we give here is relatively elementary (and does not require to use the elaborate notion of nef adelic line bundle) and allows to have explicit constants in the inequality.

Proof. Fix $0 < \varepsilon < \widehat{\mathcal{H}}_{f,\mathcal{M}}^{(m)}(\mathcal{Y})$ and $C \geq 1$. Take $n \geq 1$ such that $d^n \left(\widehat{\mathcal{H}}_{f,\mathcal{M}}^{(m)}(\mathcal{Y}) - \varepsilon/2\right) > 1$ and $C \leq d^n\varepsilon/4$. Choose integers $M, N \geq 1$ such that

$$N\left(d^n \widehat{\mathcal{H}}_{f,\mathcal{M}}^{(m)}(\mathcal{Y}) - C\right) > M \geq N d^n \left(\widehat{\mathcal{H}}_{f,\mathcal{M}}^{(m)}(\mathcal{Y}) - \varepsilon/2\right).$$

We use Lemma 1.8: increasing n if necessary, if $\mathcal{Y}_n^{(m)} := (F_n^{(m)})_*(\psi_n^{(m)})^*\mathcal{Y}^{[m]}$, we deduce from Lemma 1.9 that the quantity

$$\frac{\left(\{\mathcal{Y}_n^{(m)}\} \cdot \left(N c_1(\mathcal{L}^{[m]})\right)^{\dim \mathcal{Y}^{[m]}}\right)}{\dim \mathcal{Y}^{[m]} \left(\{\mathcal{Y}_n^{(m)}\} \cdot \left(N c_1(\mathcal{L}^{[m]})\right)^{\dim \mathcal{Y}^{[m]}-1} \cdot \left(M c_1(\pi_{[m]}^*\mathcal{M})\right)\right)}$$

is bounded from below by $\frac{N}{M}\left(d^n \widehat{\mathcal{H}}_{f,\mathcal{M}}^{(m)}(\mathcal{Y}) - C\right) > 1$. Let $\mathcal{Y}_n := (\psi_n^{(m)})^*(\mathcal{Y}^{[m]})$, $\mathcal{L}_n := \left((F_n)^*\mathcal{L}^{[m]}\right)|_{\mathcal{Y}_n}$, and $\mathcal{M}_n := \left((\pi_{[m]} \circ \psi_n^{(m)})^*\mathcal{M}\right)|_{\mathcal{Y}_n}$. By construction, the line bundles $\mathcal{L}_n$ and $\mathcal{M}_n$ are nef on $\mathcal{Y}_n$ and the above inequality implies

$$\left((N\mathcal{L}_n)^{\dim \mathcal{Y}_n}\right) > \dim \mathcal{Y}_n \left((M\mathcal{M}_n) \cdot (N\mathcal{L}_n)^{\dim \mathcal{Y}_n - 1}\right)$$

by the projection formula. We thus can apply Siu's bigness criterion [Laz, Theorem 2.2.15] and find that $N\mathcal{L}_n - M\mathcal{M}_n$ is a big line bundle on $\mathcal{Y}_n$. In particular, there exist $\ell \geq 1$ and a non-empty Zariski open set $\mathcal{U}_n \subset \mathcal{Y}_n$ such that for any $x \in \mathcal{U}_n(\bar{\mathbb{Q}})$,

$$h_{\mathcal{Y}_n, \ell(N\mathcal{L}_n - M\mathcal{M}_n)}(x) \geq -C_1$$

for some constant C_1 depending only on n. Now we use successively functorial properties of Weil height functions, see e.g. [HS]. First, we find that for any $y \in \mathcal{U}_n(\bar{\mathbb{Q}})$,

$$h_{\mathcal{Y}_n,\ell(N\mathcal{L}_n - M\mathcal{M}_n)}(y) = \ell\left(Nh_{\mathcal{Y}^{[m]},\mathcal{L}^{[m]}}(F^n(y)) - Mh_{S,\mathcal{M}}(\pi_{[m]} \circ \psi_n(y)\right) + O(1).$$

Since $F_n = (f^{[m]})^n \circ \psi_n^{(m)}$ on the non-empty Zariski open set $\mathcal{U}_n \cap (\psi_n^{(m)})^{-1}((\mathcal{Y}^{[m]})^0)$, since ψ_n is an isomorphism from $\mathcal{U}_n \cap (\psi_n^{(m)})^{-1}((\mathcal{Y}^{[m]})^0)$ to its image $\mathcal{U}^1 := \psi_n^{(m)}(\mathcal{U}_n) \cap (\mathcal{Y}^{[m]})^0$, and since $\mathcal{Y}^{[m]}$ is a subvariety of $\mathcal{X}^{[m]}$ we deduce that, for any $x \in \mathcal{U}^1(\bar{\mathbb{Q}})$,

$$h_{\mathcal{Y}_n,\ell(N\mathcal{L}_n - M\mathcal{M}_n)}(\psi_n^{-1}(x)) = \ell\left(Nh_{\mathcal{X}^{[m]},\mathcal{L}^{[m]}}(f^n(x)) - Mh_{S,\mathcal{M}}(\pi_{[m]}(x))\right) + O(1).$$

In particular, the above gives

$$\begin{aligned} h_{\mathcal{X}^{[m]},\mathcal{L}^{[m]}}((f^{[m]})^n(x)) &\geq \frac{M}{N} h_{S,\mathcal{M}}(\pi_{[m]}(x)) - C_2 \\ &\geq d^n\left(\widehat{\mathcal{H}}_{f,\mathcal{M}}^{(m)}(\mathcal{Y}) - \varepsilon/2\right) h_{S,\mathcal{M}}(\pi_{[m]}(x)) - C_2, \end{aligned}$$

for any $x \in \mathcal{U}(\bar{\mathbb{Q}})$, where C_2 is a constant depending on n. This rewrites as

$$\frac{1}{d^n} h_{\mathcal{X}^{[m]},\mathcal{L}^{[m]}}((f^{[m]})^n(x)) \geq \left(\widehat{\mathcal{H}}_{f,\mathcal{M}}^{(m)}(\mathcal{Y}) - \varepsilon/2\right) h_{S,\mathcal{M}}(\pi_{[m]}(x)) - C_3,$$

for any $x \in \mathcal{U}^1(\bar{\mathbb{Q}})$, where C_3 depends on n. We now use an estimate of Call and Silverman [CS, Theorem 3.1]: there is a constant $C_4 > 0$ depending only on $(\mathcal{X}, f, \mathcal{L})$ and $\mathcal{M}$ such that for any $x \in \mathcal{X}^0(\bar{\mathbb{Q}})$,

$$\left|\widehat{h}_{f_{\pi(x)}}(x) - h_{\mathcal{X},\mathcal{L}}(x)\right| \leq C_4\left(h_{S,\mathcal{M}}(\pi(x)) + 1\right).$$

By functorial properties of heights

$$h_{\mathcal{X}^{[m]},\mathcal{L}^{[m]}}(x) = \sum_{j=1}^{m} h_{\mathcal{X},\mathcal{L}}(x_j) + O(1), \quad x = (x_1, \ldots, x_m) \in \mathcal{X}^{[m]}(\bar{\mathbb{Q}}),$$

and the construction of the canonical height gives

$$\widehat{h}_{f^{[m]}_{\pi_{[m]}(x)}}(x) = \sum_{i=1}^{m} \widehat{h}_{f_{\pi(x_i)}}(x_i), \quad x = (x_1, \ldots, x_m) \in \mathcal{X}^{[m]}(\bar{\mathbb{Q}}),$$

since $\pi_{[m]}(x) = \pi(x_i)$ for any i by construction. Applying this inequality to $(f^{[m]})^n(x)$ and using that $\widehat{h}_{f_{\pi(x_i)}}(f^n(x_i)) = d^n \widehat{h}_{f_{\pi(x_i)}}(x_i)$ for any i, we find

$$\sum_{j=1}^{m} \widehat{h}_{f_{\pi(x_j)}}(x_j) \geq \left(\widehat{\mathcal{H}}_{f,\mathcal{M}}(\mathcal{Y}) - \frac{\varepsilon}{2} - \frac{C_4}{d^n}\right) h_{S,\mathcal{M}}(\pi_{[m]}(x)) - C_3 - \frac{C_4}{d^n},$$

for any $x = (x_1, \ldots, x_m) \in \mathcal{U}^1(\bar{\mathbb{Q}})$. Up to increasing n, we can assume $C_4 \leq d^n\varepsilon/2$, which gives the expected inequality. □

As an immediate application of Theorem 5.4, we have

Corollary 5.5. *Fix $m \geq \dim S$ and assume $\mathrm{Vol}_f^{(m)}(\mathcal{Y}) > 0$. Let $\mathcal{M}$ be any ample $\mathbb{Q}$-line bundle on S of volume 1 and let $0 < \varepsilon < \widehat{\mathcal{H}}_{f,\mathcal{M}}^{(m)}(\mathcal{Y})$. There is a non-empty Zariski open*

subset $\mathcal{U} \subset \mathcal{Y}^0$ and a constant $C \geq 1$ depending only on $(\mathcal{X}, f, \mathcal{L})$, $\mathcal{Y}$, $\mathcal{M}$, m and ε such that

$$h_{S,\mathcal{M}}(\pi(x)) \leq \frac{m}{\widehat{\mathcal{H}}^{(m)}_{f,\mathcal{M}}(\mathcal{Y}) - \varepsilon} \widehat{h}_{f_{\pi(x)}}(x) + C, \quad x \in \mathcal{U}(\bar{\mathbb{Q}}).$$

Proof. Fix $\mathcal{M}$ and $0 < \varepsilon < \widehat{\mathcal{H}}^{(m)}_{f,\mathcal{M}}(\mathcal{Y})$ and let $\mathcal{B}$ be the set of points $x \in \mathcal{Y}$ such that

$$h_{S,\mathcal{M}}(\pi(x)) > \frac{m}{\widehat{\mathcal{H}}^{(m)}_{f,\mathcal{M}}(\mathcal{Y}) - \varepsilon} \widehat{h}_{f_{\pi(x)}}(x) + C$$

where C is the constant given by Theorem 5.4. We deduce that

$$\forall (x_1, \ldots, x_m) \in \mathcal{B}^{[m]},\ h_{S,\mathcal{M}}(\pi_{[m]}(x)) > \frac{1}{\widehat{\mathcal{H}}^{(m)}_{f,\mathcal{M}}(\mathcal{Y}) - \varepsilon} \sum_{j=1}^{m} \widehat{h}_{f_{\pi(x_j)}}(x_j) + C,$$

so that $\mathcal{B}^{[m]}$ is necessarily contained in a strict Zariski closed subset of $\mathcal{Y}^{[m]}$ by Theorem 5.4, whence $\mathcal{B}$ is contained in a strict Zariski closed subset of $\mathcal{Y}$. □

5.4. **General dynamical heights as moduli heights.** Let $(\mathcal{X}, f, \mathcal{L}, \mathcal{Y})$ be a dynamical pair of degree $d \geq 2$ parametrized by a smooth projective variety S, with regular part $S^0_{\mathcal{Y}}$, all defined over a number field $\mathbb{K}$.

When Z is a subvariety of $S^0_{\mathcal{Y}}$, we let $\mathcal{Y}_Z := \pi|_{\mathcal{Y}}^{-1}(Z)$ and we define $(\mathcal{X}_Z, f_Z, \mathcal{L}_Z)$ as the family of polarized endomorphisms induced by restriction of $(\mathcal{X}, f, \mathcal{L})$ to $\mathcal{X}_Z := \pi^{-1}(Z)$.

We prove here the following

Theorem 5.6. *let $(\mathcal{X}, f, \mathcal{L})$, S and $\mathcal{Y}$ be all defined over a number field $\mathbb{K}$. Fix an embedding $\iota : \mathbb{K} \to \mathbb{C}$ for which we define the different bifurcation currents and assume that $\mu_{f,\mathcal{Y}} \neq 0$. Then, there is a non-empty Zariski open subset $U \subset S^0_{\mathcal{Y}}$ such that for any ample height h on U, there are constants $C_1, C_2 > 0$ and $C_3, C_4 \in \mathbb{R}$ such that*

$$C_1 h(t) + C_3 \leq \widehat{h}_{f_t}(Y_t) \leq C_2 h(t) + C_4 \quad \text{for all } \ t \in U(\bar{\mathbb{Q}}).$$

Moreover, for any archimedean place $v \in M_{\mathbb{K}}$, any irreducible component Z of $S^0_{\mathcal{Y}} \setminus U$ satisfies $T^{(\dim Z)}_{f,\mathcal{Y},v} \wedge [Z] = 0$.

We are now in position to prove Theorem A.

Proof of Theorem A. We work at the archimedean place of $\mathbb{Q}$. It follows from Theorem C that the bifurcation measure $\mu_{f,\mathrm{Crit}}$ is non-zero on $\mathscr{M}^k_d(\mathbb{C})$, whence it is sufficient to apply Theorem 5.6 to conclude the proof of Theorem A. □

The proof of Theorem 5.6 splits into two distinct parts that are summarized in two Propositions below. We first use Zhang's inequalities over number fields to deduce the following from Corollary 5.5:

Proposition 5.7. *Let $\mathcal{M}$ be an ample $\mathbb{Q}$-line bundle on S of volume 1 and assume $\mathrm{Vol}^{(\dim S)}_f(\mathcal{Y}) > 0$. There are constants $C_1 > 0$ and $C_2 \geq 1$ depending only on $(\mathcal{X}, f, \mathcal{L}, \mathcal{Y})$ and $\mathcal{M}$ and a non-empty Zariski open subset $V \subset S^0_{\mathcal{Y}}$ defined over $\bar{\mathbb{Q}}$ such that*

$$h_{S,\mathcal{M}}(t) \leq C_1 \widehat{h}_{f_t}(Y_t) + C_2, \quad t \in V(\bar{\mathbb{Q}}).$$

Moreover, for any irreducible component Z of $S^0_{\mathcal{Y}} \setminus V$, we have $\mathrm{Vol}_{f_Z}(\mathcal{Y}_Z) = 0$.

Proof. Let $q := \dim Y_\eta$. Fix $0 < \varepsilon < \widehat{\mathcal{H}}_{f,\mathcal{M}}^{(\dim S)}(\mathcal{Y})$. Let $\mathcal{U}$ be the Zariski open subset in Corollary 5.5, let V be the set of $t \in \pi(\mathcal{U})$ so that $U_t := \mathcal{U} \cap Y_t$ is non-empty a Zariski open subset of Y_t. The set V is a Zariski open subset of S^0 and for any $t \in V(\bar{\mathbb{Q}})$, we have

$$h_{S,\mathcal{M}}(t) \leq \frac{\dim S}{\widehat{\mathcal{H}}_{f,\mathcal{M}}(\mathcal{Y}) - \varepsilon} \left(\widehat{h}_{f_t}(x) + C \right), \quad x \in U_t(\bar{\mathbb{Q}}).$$

Taking the infimum of $\widehat{h}_{f_t}(x)$ over $x \in U_t(\bar{\mathbb{Q}})$ and using Zhang's inequalities (13) gives

$$h_{S,\mathcal{M}}(t) \leq \frac{\dim S}{\widehat{\mathcal{H}}_{f,\mathcal{M}}(\mathcal{Y}) - \varepsilon} \left((q+1)\widehat{h}_{f_t}(Y_t) + C \right).$$

This is the wanted inequality, but we may have restricted too much the open set.

To conclude, we can proceed exactly the same way on any irreducible component Z of $S^0_{\mathcal{Y}} \setminus V$, where $\mathrm{Vol}_{f_Z}^{(\dim Z)}(\mathcal{Y}_Z) > 0$. In finitely many steps, we end with the expected result. □

We now use another description of the height $h_{\bar{\mathcal{L}}}(Y_t)$, when $t \in S^0_{\mathcal{Y}}(\bar{\mathbb{Q}})$, using Chow forms as in [H]. The next is probably well-known, but we include a proof for the sake of completeness.

Lemma 5.8. *Let S be a projective variety, let $\pi : \mathcal{Y} \to S$ be a surjective morphism, both defined over a number field $\mathbb{K}$. Let $\bar{\mathcal{L}}$ be a relatively ample line bundle on $\mathcal{Y}$. Let $S^0_{\mathcal{Y}} \subseteq S$ be a Zariski open set such that π is flat over $S^0_{\mathcal{Y}}$.*

For any ample line bundle $\mathcal{M}$ on S, there are constants $C_1, C_2 > 0$ such that

$$h_{\bar{\mathcal{L}}}(Y_t) \leq C_1 h_{S,\mathcal{M}}(t) + C_2, \quad t \in S^0_{\mathcal{Y}}(\bar{\mathbb{Q}}).$$

Proof. Up to replacing $\mathcal{L}$ by a large multiple and up to changing the metrization on $\mathcal{L}$, we may assume that there is an embedding $\iota : \mathcal{Y} \hookrightarrow \mathbb{P}^N_S$ such that $\bar{\mathcal{L}} = \iota^* \bar{\mathcal{O}}_{\mathbb{P}^N}(1)$, so that

$$h_{\bar{\mathcal{L}}}(Y_t) = h_{\mathbb{P}^N}(\iota_*(Y_t)), \quad \text{for all } t \in S^0_{\mathcal{Y}}(\bar{\mathbb{Q}}),$$

where $h_{\mathbb{P}^N}$ is the naive height function on $\mathbb{P}^N$. This is where Chow forms are used, to give a different description of $h_{\mathbb{P}^N}(Y_t)$, which makes easier the expected inequality.

For any irreducible subvariety $Y \subset \mathbb{P}^N$ of dimension $q \geq 1$, in the Grassmannian $G(N-k-1, N)$ of linear subspaces of dimension $N-k-1$ of $\mathbb{P}^N$, the set

$$\mathcal{Z}_Y := \{V \in G(N-k-1, N)\,;\ V \cap Y \neq \varnothing\}$$

is an irreducible hypersurface. Moreover, in the Plücker coordinates, we have $\mathcal{Z}_Y = \{\mathcal{R}_Y = 0\}$, where $\mathcal{R}_Y$ is a homogeneous polynomial satisfying the following properties, see, e.g., [DS1, H]:

(1) if Y is defined over $\bar{\mathbb{Q}}$, then $\mathcal{R}_Y$ is also defined over $\bar{\mathbb{Q}}$,
(2) $\deg(\mathcal{R}_Y) = \deg(Y)$,
(3) $h_{\mathbb{P}^N}(Y) = h([a_0 : \cdots : a_M])$, where $a_0, \ldots, a_M$ are the coefficients of $\mathcal{R}_Y$.

Coming back to our situation, the above gives

$$h_{\bar{\mathcal{L}}}(Y_t) = h([a_0(t) : \cdots : a_M(t)]), \quad t \in S^0_{\mathcal{Y}}(\bar{\mathbb{Q}}).$$

We now observe that the map $A : t \in S^0_{\mathcal{Y}} \mapsto [a_0(t) : \cdots : a_M(t)] \in \mathbb{P}^M$ is regular and defined over $\bar{\mathbb{Q}}$, i.e. $A \in \bar{\mathbb{Q}}[S^0_{\mathcal{Y}}]$. This observation is true by construction of the Chow form, see, e.g., [SZ, §3]. □

As an application of Call and Silverman's fundamental work [CS] and from Lemma 5.8, we prove Theorem 5.6:

Proof of Theorem 5.6. The left hand side inequality is proved in Proposition 5.7. We now prove the right hand side inequality. Fix any closed point $t \in S^0(\bar{\mathbb{Q}})$. By Zhang's inequality (13), if (x_j) is a generic sequence of closed points of $Y_t(\bar{\mathbb{Q}})$, we have

$$\widehat{h}_{f_t}(Y_t) \leq \liminf_{j\to\infty} \widehat{h}_{f_t}(x_j) \quad \text{and} \quad \frac{1}{q+1} \liminf_{j\to\infty} h_{\bar{\mathcal{L}}}(x_j) \leq h_{\bar{\mathcal{L}}}(Y_t).$$

We now apply [CS, Theorem 3.1]: there exists constants $C, C' > 0$ depending only on $(\mathcal{X}, f, \mathcal{L}, \mathcal{Y})$ and on $\mathcal{M}$ such that

$$\left|\widehat{h}_{f_t}(x) - h_{\bar{\mathcal{L}}}(x)\right| \leq C h_{\mathcal{M}}(t) + C',$$

for all $x \in Y_t(\bar{\mathbb{Q}})$. The above implies

$$\widehat{h}_{f_t}(Y_t) \leq (q+1) h_{\bar{\mathcal{L}}}(Y_t) + C h_{\mathcal{M}}(t) + C'.$$

The conclusion follows from Lemma 5.8 above. □

6. Two dynamical equidistribution results

6.1. **Good height functions on quasi-projective varieties.** Let V be a smooth quasi-projective variety defined over a number field $\mathbb{K}$ and let $\mathbb{K} \hookrightarrow \mathbb{C}$ be an embedding and let $h : V(\bar{\mathbb{Q}}) \to \mathbb{R}$ be a function. A sequence $(F_i)_i$ of Galois-invariant finite subsets of $V(\bar{\mathbb{Q}})$ is

- *generic* if for any subvariety $Z \subset V$ defined over $\mathbb{K}$, there is i_0 such that $F_i \cap Z = \varnothing$ for $i \geq i_0$, and
- *h-small* if $h(F_i) := \frac{1}{\#F_i} \sum_{x\in F_i} h(x) \to 0$, as $i \to \infty$.

As in [Ga3], we say h is a *good height* at the complex place if for any $n \geq 0$, there is a projective model X_n of V together with a birational morphism $\psi_n : X_n \to X_0$ which is an isomorphism above V and a big and nef $\mathbb{Q}$-line bundle L_n on X_n endowed with an adelic semi-positive continuous metrization $\bar{L}_n$, such that the following holds :

(1) For any generic h-small sequence $(F_i)_i$ of Galois-invariant finite subsets of $V(\bar{\mathbb{Q}})$, the sequence $\varepsilon_n(\{F_i\}_i) := \limsup_i h_{\bar{L}_n}(\psi_n^{-1}(F_i)) - h_{\bar{L}_n}(X_n)$ satisfies $\varepsilon_n(\{F_i\}) \to 0$ as $n \to \infty$,
(2) the sequence of volumes $\mathrm{vol}(L_n)$ converges to $\mathrm{vol}(h) > 0$ as $n \to \infty$ and if $c_1(\bar{L}_n)$ is the curvature form of $\bar{L}_n$ on $X_n(\mathbb{C})$, then the sequence of finite measures $\left(\mathrm{vol}(L_n)^{-1}(\psi_n)_* c_1(\bar{L}_n)^k\right)_n$ converges weakly on $V(\mathbb{C})$ to a probability measure μ,
(3) If $k := \dim V > 1$, for any ample line bundle M_0 on X_0 and any adelic semi-positive continuous metrization $\bar{M}_0$ on M_0, there is a constant $C \geq 0$ such that

$$\left(\psi_n^*(\bar{M}_0)\right)^j \cdot \left(\bar{L}_n\right)^{k+1-j} \leq C,$$

for any $2 \leq j \leq k+1$ and any $n \geq 0$.

We say that $\mathrm{vol}(h)$ is the *volume* of h and that μ is the measure *induced by* h over the complex numbers.

The first author proved in [Ga3, Theorem 1] the next result:

Theorem 6.1. *For any h-small and generic sequence $(F_m)_m$ of Galois-invariant finite subsets of $V(\bar{\mathbb{Q}})$, the probability measure μ_{F_m} on $V(\mathbb{C})$ which is equidistributed on F_m converges to μ in the weak sense of measures, i.e. for any $\varphi \in \mathscr{C}_c^0(V(\mathbb{C}))$, we have*

$$\lim_{m\to\infty} \frac{1}{\#F_m} \sum_{y\in F_m} \varphi(y) = \int_{V(\mathbb{C})} \varphi\, \mu.$$

6.2. **A dynamical relative equidistribution Theorem.** When $\pi : \mathcal{A} \to S$ is a family of abelian varieties defined over a number field $\mathbb{K}$, where S is a smooth projective variety, and $\mathcal{Y} \subset \mathcal{A}$ is a non-degenerate subvariety also defined over $\mathbb{K}$, Kühne [K] proposes and proves a *Relative Equidistribution Conjecture* which, in turn, says that if there is a generic sequence $\{x_i\}_i$ in $\mathcal{Y}^0(\bar{\mathbb{Q}})$ with $\widehat{h}_{\mathcal{A}}(x_i) \to 0$, then the measure μ_{x_i} on $\mathcal{Y}^0(\mathbb{C})$ equidistributed on the Galois orbit $\mathsf{O}(x_i)$ converges weakly on $\mathcal{Y}^0(\mathbb{C})$ to a given probability measure μ.

We want here to prove the next dynamical generalization of Kühne's Relative Equidistribution Conjecture:

Theorem 6.2 (Dynamical Relative Equidistribution)**.** *Let $(\mathcal{X}, f, \mathcal{L})$ be a family of polarized endomorphisms parametrized by a smooth projective variety S and let $\mathcal{Y} \subset \mathcal{X}$ be a family of subvarieties of $\mathcal{X}$. Assume $\mu_{f,\mathcal{Y}}$ is non zero on $S^0(\mathbb{C})$.*

Then for any $m \geq \dim S$ and any $\varphi \in \mathscr{C}_c^0((\mathcal{Y}^{[m]})^0(\mathbb{C}), \mathbb{R})$ and any generic and $\widehat{h}_{f^{[m]}}$-small sequence $\{F_i\}_i$ of Galois invariant subsets of $(\mathcal{Y}^{[m]})^0(\bar{\mathbb{Q}})$, we have

$$\lim_{i\to\infty} \frac{1}{\#F_i} \sum_{x\in F_i} \varphi(x) = \frac{1}{\mathrm{Vol}_f^{(m)}(\mathcal{Y})} \int_{(\mathcal{Y}^{[m]})^0(\mathbb{C})} \varphi\, \widehat{T}_{f^{[m]}}^{\dim \mathcal{Y}^{[m]}}.$$

As in section 6.1, we could have used the theory of adelic line bundles by Yuan and Zhang [YZ] (see Theorem 6.2.3 therein) and again, we do not use the full strength of their theory here.

Proof. We fix an archimedean place of $\mathbb{K}$ and a corresponding embedding $\mathbb{K} \hookrightarrow \mathbb{C}$. By Theorem 6.1, all there is to prove is that $\widehat{h}_{f^{[m]}}$ is a good height function on $(\mathcal{Y}^{[m]})^0$ and to show its induced measure on $(\mathcal{Y}^{[m]})^0(\mathbb{C})$ is indeed $\widehat{T}_{f^{[m]}}^{\dim \mathcal{Y}^{[m]}}$.

Let $\mathcal{M}$ be an ample $\mathbb{Q}$-line bundle on S of volume 1 and let $\mathcal{L}_0 := \mathcal{L}^{[m]} + \pi_{[m]}^*\mathcal{M}$. The line bundle $\mathcal{L}_0$ is ample on $\mathcal{X}^{[m]}$. Recall Call and Silverman's result [CS, Theorem 3.1] guarantees the existence of $C \geq 1$ such that

$$\left|\widehat{h}_f(x) - h_{\mathcal{X},\mathcal{L}}(x)\right| \leq C\left(h_S(\pi(x)) + 1\right),$$

for all $x \in \mathcal{X}^0(\bar{\mathbb{Q}})$. As in the proof of Theorem 5.4, using that $\widehat{h}_f \circ f = d \cdot \widehat{h}_f$, we deduce that up to changing the constant C, we have

$$\left|\widehat{h}_{f^{[m]}}(x) - \frac{1}{d^n} h_{\mathcal{X}^{[m]},\mathcal{L}^{[m]}}((f^{[m]})^n(x))\right| \leq \frac{C}{d^n}\left(h_S(\pi_{[m]}(x)) + 1\right),$$

for any $x \in (\mathcal{X}^{[m]})^0(\bar{\mathbb{Q}})$ and any $n \geq 0$. As $((f^{[m]})^n)^*\mathcal{L}_0 = d^n\mathcal{L}^{[m]} + \pi_{[m]}^*\mathcal{M}$, this implies

$$\left|\widehat{h}_{f^{[m]}}(x) - \frac{1}{d^n} h_{\mathcal{X}^{[m]},\mathcal{L}_0}((f^{[m]})^n(x))\right| \leq \frac{C}{d^n}\left(h_S(\pi_{[m]}(x)) + 2\right),$$

for any $x \in (\mathcal{X}^{[m]})^0(\bar{\mathbb{Q}})$ and any $n \geq 0$. We now use Theorem 5.4: there is a non-empty Zariski open set $\mathcal{V} \subset (\mathcal{Y}^{[m]})^0$ such that for any $x \in \mathcal{V}(\bar{\mathbb{Q}})$, we have

$$h_S(\pi_{[m]}(x)) \leq \frac{2}{\widehat{\mathcal{H}}^{(m)}_{f,\mathcal{M}}(\mathcal{Y})} \left(\widehat{h}_{f^{[m]}}(x) + 1\right).$$

We thus have a constant $C_2 > 0$ such that for any $x \in \mathcal{V}(\bar{\mathbb{Q}})$ and any $n \geq 0$,

$$\left|\widehat{h}_{f^{[m]}}(x) - \frac{1}{d^n} h_{\mathcal{X}^{[m]},\mathcal{L}^{[m]}}((f^{[m]})^n(x))\right| \leq \frac{C_2}{d^n}\left(\widehat{h}_{f^{[m]}}(x) + 1\right). \tag{14}$$

We now use Lemma 1.8: let $F_n, \psi_n : \mathcal{X}_n \to \mathcal{X}^{[m]}$ be such that $F_n = (f^{[m]})^n \circ \psi_n$ on $\psi_n^{-1}(\mathcal{X}^0)$ with ψ_n birational. We also let $\pi_n : \mathcal{X}_n \to S$ be the structure morphism induced by $\pi_{[m]}$, i.e. such that $\pi_n = \pi_{[m]} \circ \psi_n$. Choose a model metric $\bar{\mathcal{M}}$ on $\mathcal{M}$ with $h_{\bar{\mathcal{M}}} \geq 0$ on $S(\bar{\mathbb{Q}})$. We endow $\mathcal{L}$ with a metrization $\bar{\mathcal{L}}$ coming from the embedding $\iota : \mathcal{X} \hookrightarrow \mathbb{P}^N \times S$ for which (a multiple) of $\mathcal{L}$ is $\iota^*\bar{\mathcal{O}}_{\mathbb{P}^N}(1)$, where $\bar{\mathcal{O}}_{\mathbb{P}^N}(1)$ is the naive metrization. Define

$$\bar{\mathcal{L}}_0 := \bar{\mathcal{L}}^{[m]} + (\pi_{[m]})^*\bar{\mathcal{M}}.$$

We then let $\mathcal{Y}_n := \psi_n^{-1}(\mathcal{Y}^{[m]})$ and

$$\bar{\mathcal{L}}_n := \frac{1}{d^n}\left(F_n^*\bar{\mathcal{L}}_0\right)|_{\mathcal{Y}_n} = \frac{1}{d^n}\left(F_n^*\bar{\mathcal{L}}^{[m]}\right)|_{\mathcal{Y}_n} + \frac{1}{d^n}(\pi_n^*\bar{\mathcal{M}})|_{\mathcal{Y}_n}, \quad n \geq 0.$$

By construction the map F_n is a generically finite morphism. Since $\bar{\mathcal{L}}_0$ is an adelic semi-positive continuous ample line bundle, $\bar{\mathcal{L}}_n$ is thus an adelic semi-positive continuous big and nef $\mathbb{Q}$-line bundle on $\mathcal{Y}_n$. Moreover, by construction, we have

$$h_{\mathcal{Y}_n,\bar{\mathcal{L}}_n}(\psi_n^{-1}(x)) = \frac{1}{d^n} h_{\mathcal{X}^{[m]},\mathcal{L}_0}((f^{[m]})^n(x)),$$

for any $n \geq 0$ and any $x \in (\mathcal{Y}^{[m]})^0(\bar{\mathbb{Q}})$. Note also that, by construction, $h_{\mathcal{Y}_n,\bar{\mathcal{L}}_n} \geq 0$ on $\mathcal{Y}_n(\bar{\mathbb{Q}})$, so that [Ga3, Lemma 6] gives

$$h_{\bar{\mathcal{L}}_n}(\mathcal{Y}_n) \geq 0.$$

We combine this inequality with the inequality (14): this implies that for any generic sequence $\{F_i\}_i$ of Galois invariant subsets of $(\mathcal{Y}^{[m]})^0(\bar{\mathbb{Q}})$ with $\widehat{h}_{f^{[m]}}(F_i) \to 0$, we have

$$\limsup_{i\to\infty}\left(h_{\mathcal{Y}_n,\bar{\mathcal{L}}_n}(\psi_n^{-1}(F_i)) - h_{\mathcal{Y}_n,\bar{\mathcal{L}}_n}(\mathcal{Y}_n)\right) \leq \limsup_{i\to\infty} h_{\mathcal{Y}_n,\bar{\mathcal{L}}_n}(\psi_n^{-1}(F_i)) \leq 2\frac{C_2}{d^n}.$$

We now let ω and ρ be the respective curvature forms $\omega := c_1(\bar{\mathcal{L}}|_{\mathcal{Y}})$ and $\rho = c_1(\bar{\mathcal{M}})$ on $\mathcal{Y}(\mathbb{C})$ and $S(\mathbb{C})$ respectively. Then ω is a smooth form on $\mathcal{Y}(\mathbb{C})$ representing $c_1(\mathcal{L}|_{\mathcal{Y}})$, and if we denote as before $p_j : \mathcal{Y}^{[m]} \to \mathcal{Y}$ the projection onto the j-th factor of the fiber-product, the curvature form of $\bar{\mathcal{L}}_n$ satisfies as forms on $(\mathcal{Y}^{[m]})(\mathbb{C})$:

$$c_1(\bar{\mathcal{L}}_n)^{\dim \mathcal{Y}^{[m]}} = d^{-n\dim\mathcal{Y}^{[m]}}\psi_n^*\left(((f^{[m]})^n)^*\left(p_1^*(\omega) + \cdots + p_m^*(\omega) + \pi_{[m]}^*(\rho)\right)^{\dim\mathcal{Y}^{[m]}}\right)$$

so that, if $\omega_m := p_1^*(\omega) + \cdots + p_m^*(\omega)$, we have as measures on $(\mathcal{Y}^{[m]})(\mathbb{C})$:

$$(\psi_n)_*\left(c_1(\bar{\mathcal{L}}_n)^{\dim\mathcal{Y}^{[m]}}\right) = d^{-n\dim\mathcal{Y}^{[m]}}\left(((f^{[m]})^n)^*\left(\omega_m + \pi_{[m]}^*(\rho)\right)^{\dim\mathcal{Y}^{[m]}}\right).$$

Now, as $d^{-n}((f^{[m]})^n)^*\omega_m$ converges to $\widehat{T}_{f^{[m]}}$ with a uniform convergence of local potentials and as we have $((f^{[m]})^n)^*\pi_{[m]}^*(\rho) = \pi_{[m]}^*(\rho)$ by construction, the following holds in the weak sense of measures on $(\mathcal{Y}^{[m]})(\mathbb{C})$:

$$d^{-n\dim \mathcal{Y}^{[m]}}\left(((f^{[m]})^n)^*\left(\omega_m + \pi_{[m]}^*(\rho)\right)^{\dim \mathcal{Y}^{[m]}}\right) \to \widehat{T}_{f^{[m]}}^{\dim \mathcal{Y}^{[m]}}.$$

Finally, the volume of $\mathcal{L}_n$ can be computed as

$$\deg_{\mathcal{Y}_n}(\mathcal{L}_n) = \mathrm{Vol}_f^{(m)}(\mathcal{Y}) + O\left(\frac{1}{d^n}\right).$$

Indeed, by definition of $\mathcal{L}_0$ and by Lemma 1.2, we find

$$\begin{aligned}\deg_{\mathcal{Y}_n}(\mathcal{L}_n) &= \left(c_1(\mathcal{L}_n)^{\dim \mathcal{Y}^{[m]}} \cdot \{\mathcal{Y}_n\}\right) = \left(d^{-n\dim \mathcal{Y}^{[m]}}(F_n)^*c_1(\mathcal{L}_0)^{\dim \mathcal{Y}^{[m]}} \cdot \{\mathcal{Y}^{[m]}\}\right)\\ &= \left(d^{-n\dim \mathcal{Y}^{[m]}}(F_n)^*c_1(\mathcal{L}^{[m]})^{\dim \mathcal{Y}^{[m]}} \cdot \{\mathcal{Y}^{[m]}\}\right) + O\left(\frac{1}{d^n}\right)\\ &= \mathrm{Vol}_f^{(m)}(\mathcal{Y}) + O\left(\frac{1}{d^n}\right).\end{aligned}$$

Our assumption that $\mathrm{Vol}_f^{(m)}(\mathcal{Y}) > 0$ thus implies $\lim_{n\to\infty} \deg_{\mathcal{Y}_n}(\mathcal{L}_n) = \mathrm{Vol}_f^{(m)}(\mathcal{Y}) > 0$.

To prove that $\widehat{h}_{f^{[m]}}$ is a good height function on $(\mathcal{Y}^{[m]})^0(\bar{\mathbb{Q}})$, the last thing to check is condition (3) introduced in Section 6.1. Let $\pi_n : \mathcal{Y}_n \to S$ be the morphism induced by $\pi_{[m]} : \mathcal{Y}^{[m]} \to S$.

Let $\bar{M}_0$ be an ample adelic semi-positive continuous line bundle on $\mathcal{Y}_0$. Then $\psi_n^*\bar{M}_0$ is a big and nef $\mathbb{Q}$-line bundle on $\mathcal{Y}_n$ and $\psi_n^*\bar{M}_0$ is a semi-positive adelically metrized line bundle on $\mathcal{Y}_n$. Let $\bar{\mathcal{E}} := d^{-1}F_1^*\bar{\mathcal{L}} - \psi_1^*\bar{\mathcal{L}}$. Then, there is $\bar{D}$ effective on S such that $-d^{-1}\pi_1^*\bar{D} \le \bar{\mathcal{E}} \le d^{-1}\pi_1^*\bar{D}$.

By construction, we can assume there is a birational morphism $\phi_n : \mathcal{Y}_{n+1} \to \mathcal{Y}_n$ with $\pi_n \circ \phi_{n+1} = \pi_{n+1}$ and that $\psi_{n+1} = \psi_n \circ \phi_{n+1}$. Without loss of generality, we can also assume $\psi_n = \phi_1 \circ \cdots \circ \phi_n$ and there is a morphism $g_n : \mathcal{Y}_{n+1} \to \mathcal{Y}_1$ such that

$$\phi_1 \circ g_n = F_n \circ \phi_{n+1} \text{ and } F_1 \circ g_n = F_{n+1} \text{ on } \mathcal{Y}_{n+1}.$$

We have $d^{-n}g_n^*(\bar{\mathcal{E}}) \le d^{-(n+1)}g_n^*\pi_1^*\bar{D} = d^{-(n+1)}\pi_{n+1}^*\bar{D}$. In particular, one sees that

$$\begin{aligned}\bar{\mathcal{L}}_{n+1} - \phi_{n+1}^*\bar{\mathcal{L}}_n &= \frac{1}{d^n}g_n^*\left(\bar{\mathcal{E}}\right) + \left(\frac{1}{d^{n+1}} - \frac{1}{d^n}\right)\pi_{n+1}^*(\bar{\mathcal{M}})\\ &\le \frac{1}{d^{n+1}}\pi_{n+1}^*(\bar{D} + \bar{\mathcal{M}}).\end{aligned}$$

Whence $\bar{\mathcal{L}}_{n+1} \le \phi_{n+1}^*(\bar{\mathcal{L}}_n + d^{-(n+1)}\pi_n^*(\bar{D} + \bar{\mathcal{M}}))$. An immediate induction gives

$$\bar{\mathcal{L}}_n \le \psi_n^*\left(\bar{\mathcal{L}}_0 + \frac{d}{d-1}\pi_{[m]}^*(\bar{D} + \bar{\mathcal{M}})\right). \tag{15}$$

Let $P := \dim \mathcal{Y}^{[m]}$ and pick $0 \le \ell \le P+1$. For all $n \ge 0$, (15) gives

$$\left(\left(\psi_n^*(\bar{M}_0)\right)^\ell \cdot \left(\bar{\mathcal{L}}_n\right)^{P+1-\ell} \middle| \mathcal{Y}_n \right) \le \left(\left(\psi_n^*(\bar{M}_0)\right)^\ell \cdot \psi_n^* \left(\bar{\mathcal{L}}_0 + \frac{d}{d-1}\pi_{[m]}^*(\bar{D}+\bar{\mathcal{M}}) \right)^{P+1-\ell} \middle| \mathcal{Y}_n \right)$$
$$\le \left(\left(\bar{M}_0\right)^\ell \cdot \left(\bar{\mathcal{L}}_0 + \frac{d}{d-1}\pi_{[m]}^*(\bar{D}+\bar{\mathcal{M}}) \right)^{P+1-\ell} \middle| \mathcal{Y}^{[m]} \right),$$

where we used the projection formula and that $(\psi_n)_*(\mathcal{Y}_n) = \mathcal{Y}^{[m]}$. This proves hypothesis (3) of section 6.1 is satisfied as the last quantity is independent of n and the proof of Theorem 6.1 is complete. □

6.3. **Parametric equidistribution.** For any finite Galois invariant subset $F \subset S^0(\bar{\mathbb{Q}})$, we define $h_{f,\mathcal{Y}}(F)$ as

$$h_{f,\mathcal{Y}}(F) := \frac{1}{\#F} \sum_{t \in F} \widehat{h}_{f_t}(Y_t).$$

As usual, we say a sequence F_i of finite Galois invariant subsets of $S^0(\bar{\mathbb{Q}})$ is *$h_{f,\mathcal{Y}}$-small* if $h_{f,\mathcal{Y}}(F_i) \to 0$. Using Zhang inequalities over number fields, we can deduce the following from Theorem 6.1.

Corollary 6.3. *Let $(\mathcal{X}, f, \mathcal{L}, \mathcal{Y})$ be a dynamical pair parametrized by a smooth projective variety S with regular part S^0, all defined over a number field $\mathbb{K}$. Assume $\mathrm{Vol}_f^{(\dim S)}(\mathcal{Y}) > 0$. Assume also there is a generic and $h_{f,\mathcal{Y}}$-small sequence $\{F_i\}_i$ of finite Galois invariant subsets of $S^0(\bar{\mathbb{Q}})$. Then for any $\varphi \in \mathscr{C}_c^0(S^0(\mathbb{C}), \mathbb{R})$, we have*

$$\lim_{i\to\infty} \frac{1}{\#F_i} \sum_{t \in F_i} \varphi(t) = \int_{S^0(\mathbb{C})} \varphi \, \frac{\mu_{f,\mathcal{Y}}}{\mathrm{Vol}_f(\mathcal{Y})}.$$

Proof. Fix $m \ge \dim S$ and fix i and let $t \in F_i$. Zhang's inequalities (13) imply there exists a generic sequence $\{x_j^{(t)}\}_j$ of $Y_t^{[m]}(\bar{\mathbb{Q}})$ such that we have

$$\limsup_{j\to\infty} \widehat{h}_{f^{[m]}}(x_j^{(t)}) \le (q+1)\widehat{h}_{f_t^{[m]}}(Y_t^{[m]}) \ .$$

For any i, j, we define a finite Galois invariant subset Z_j^i of $(\mathcal{Y}^{[m]})^0(\bar{\mathbb{Q}})$ by letting

$$Z_j^i := \bigcup_{t \in F_i} \mathsf{O}(x_j^{(t)}).$$

By the above, and by Lemma 5.1, we deduce that

$$\liminf_{j\to\infty} \widehat{h}_{f^{[m]}}(Z_j^i) \le (q+1) h_{f^{[m]},\mathcal{Y}^{[m]}}(F_i) = m(q+1) \cdot h_{f,\mathcal{Y}}(F_i).$$

Take $\varepsilon_i > 0$ such that $\varepsilon_i \to 0$ as $i \to \infty$ and such that $(q+1)^2 m \cdot h_{f,\mathcal{Y}}(F_i) \le \varepsilon_i$ for any i. For any $i \ge 1$, there is an infinite sequence $(j_n(i+1))_n$, extracted from $(j_n(i))_n$ such that, for any $n \ge 0$, we have $\widehat{h}_{f^{[m]}}(Z^i_{j_n(i)}) \le 2\varepsilon_i$. We deduce there exists a sequence $\{Z_i\}_i$ of finite Galois invariant finite subsets of $(\mathcal{Y}^{[m]})^0(\bar{\mathbb{Q}})$ such that $\pi_{[m]}(Z_i) = F_i$ and such that for any $t \in F_i$, we have $\mathsf{O}(x^{(t)}_{j(i)}) \subset \pi_{[m]}^{-1}\{t\}$ for some $j(i) \ge j_0(i)$. Moreover, by construction we can choose Z_i generic, and we have

$$0 \le \frac{1}{\#Z_i} \sum_{x \in Z_i} \widehat{h}_{f^{[m]}}(x) \le 2\varepsilon_i.$$

As $\varepsilon_i \to 0$ and $\{Z_i\}_i$ is generic, Theorem 6.2 implies

$$\frac{1}{\#Z_i}\sum_{x\in Z_i}\delta_{x,v} \to \mu_v,$$

where μ_v is a probability measure on $(\mathcal{Y}^{[m]})_v^{0,\mathrm{an}}$ which satisfies

$$(\pi_{[m]})_*(\mu_v) = \mu_{f,[\mathcal{Y}],v}(S_v^{0,\mathrm{an}})^{-1}\mu_{f,[\mathcal{Y}],v},$$

when v is archimedean. Let $\nu_v := (\pi_{[m]})_*(\mu_v)$ and take $\varphi \in \mathscr{C}_v^0(S_v^{0,\mathrm{an}},\mathbb{R})$. Then

$$\frac{1}{\#F_i}\sum_{t\in F_i}\varphi(t) = \frac{1}{\#F_i}\sum_{t\in F_i}\frac{1}{\#\mathsf{O}(x_{j(i)}^{(t)})}\sum_{x\in\mathsf{O}(x_{j(i)}^{(t)})}\varphi(\pi_{[m]}(x)) = \frac{1}{\#Z_i}\sum_{x\in Z_i}\varphi(\pi_{[m]}(x)).$$

We now use that $\nu_v = (\pi_{[m]})_*(\mu_v)$ so that

$$\int_{S_v^{0,\mathrm{an}}}\varphi\nu_v = \int_{(\mathcal{Y}^{[m]})_v^{0,\mathrm{an}}}(\varphi\circ\pi_{[m]})\mu_v.$$

Finally, if v is archimedean, since $(\pi_{[m]})_*(\mu_v) = \left(\mu_{f,[\mathcal{Y}],v}(S_v^{0,\mathrm{an}})\right)^{-1}\cdot\mu_{f,\mathcal{Y},v}$, we have $\nu_v = \left(\mu_{f,\mathcal{Y},v}(S_v^{0,\mathrm{an}})\right)^{-1}\cdot\mu_{f,\mathcal{Y},v}$ and the proof is complete. □

7. Sparsity and uniformity: proof of the main results

We are now interested in applying all the above results in two specific situations, where we study the variations of the dynamics of the critical set.

7.1. Sparsity of PCF maps of $\mathbb{P}^k$.

We focus the universal family

$$f : \mathbb{P}^k_{\mathscr{M}_d^k} \longrightarrow \mathbb{P}^k_{\mathscr{M}_d^k}$$

which is a family $(\mathbb{P}^k_S, f, \mathcal{O}_{\mathbb{P}^k}(1))$ of degree d endomorphisms of $\mathbb{P}^k$ parametrized by a projective model S of $\mathscr{M}_d^k$ with regular part $\mathscr{M}_d^k$ – if we follow the notations introduced above – which is defined over $\mathbb{Q}$, see § 2.1.

The critical variety $\mathrm{Crit}(f) \subsetneq \mathbb{P}^k_S$ satisfies $\pi(\mathrm{Crit}(f)) = S$, where $\pi : \mathbb{P}^k_S \to S$ is the canonical projection, and $\pi|_{\mathrm{Crit}(f)}$ is flat and projective over a Zariski open subset $S^0 \subseteq \mathscr{M}_d^k$. Moreover, for any $t \in S^0$, the fiber $\mathrm{Crit}(f_t) = \pi|_{\mathrm{Crit}(f)}^{-1}(t)$ is the critical locus of f_t. Moreover, up to reducing the open set S^0, we can assume $\mathrm{Crit}(f_t)$ is irreducible for all $t \in S^0$.

We are now in position to prove Theorem B.

Proof of Theorem B. By [PST], there exists a closed subvariety $Z \subsetneq \mathrm{End}_d^k$ such that, for any field, the natural projection $\Pi : \mathrm{End}_d^k \setminus Z \to \mathscr{M}_d^k \setminus \Pi(Z)$ is a principal $PGL(k+1)$-bundle, whence the set of PCF maps $f \in \mathrm{End}_d^k$ is Zariski dense in End_d^k if and only if their conjugacy classes are Zariski dense in $\mathscr{M}_d^k$ (see also [Si1]). We again work at the archimedean place of $\mathbb{Q}$. To prove Theorem B, we proceed by contradiction. Assume PCF parameters are Zariski dense in $\mathscr{M}_d^k$. By Proposition 5.7, the critical height dominates an ample height outside of a proper closed subvariety $V \subsetneq \mathscr{M}_d^k$ of $\mathscr{M}_d^k$, whence all classes of PCF maps $[f] \in U := \mathscr{M}_d^k \setminus V$ are of bounded height. By definition, all irreducible components of the set $\{[f] \in U(\mathbb{C}) :\ f \text{ is PCF}\}$ are defined over $\bar{\mathbb{Q}}$. In particular, if such

a component Z had dimension $\ell > 0$, then we would have $h_{\mathrm{Crit}} \equiv 0$ on $Z(\bar{\mathbb{Q}})$, whence $h_{\mathscr{M}_d^k} \leq C$ on $Z(\bar{\mathbb{Q}})$. This is a contradiction. Whence the PCF classes $[f] \in U(\mathbb{C})$ are countable and defined over $\bar{\mathbb{Q}}$.

We thus can find a generic sequence $(t_n)_{n\in\mathbb{N}}$ of PCF parameters $t_n \in \mathscr{M}_d^k(\bar{\mathbb{Q}})$. Let μ_n be the measure of $\mathscr{M}_d^k(\mathbb{C})$ equidistributed on the Galois orbit $\mathsf{O}(t_n)$ of t_n. By the parametric equidistribution Theorem (see Corollary 6.3), we have

$$\mu_n := \frac{1}{\mathrm{Card}(\mathsf{O}(t_n))} \sum_{t\in\mathsf{O}(t_n)} \delta_t \to \mu_{f,\mathrm{Crit}}, \quad \text{as } n \to \infty.$$

In particular, in the analytic topology of $\mathscr{M}_d^k(\mathbb{C})$, the support of $\mu_{f,\mathrm{Crit}}$ is accumulated by PCF classes. In particular, if $\mathrm{supp}(\mu_{f,\mathrm{Crit}})$ has non-empty interior, then PCF parameters are dense in it. We use again Theorem C: there exists a non-empty open set $\Omega \subset \mathscr{M}_d^k(\mathbb{C})$ contained in $\mathrm{supp}(\mu_{f,\mathrm{Crit}})$ which contains no PCF parameters. This is a contradiction. □

7.2. **Height gap and uniformity for regular maps of the affine space.** In this section, we focus on the case when $\mathcal{X} = \mathbb{P}^k \times S$ and where there is a hyperplane $H_\infty \subset \mathbb{P}^k$ such that $f_t^{-1}(H_\infty) = H_\infty$ for all $t \in S^0$. We call such a family a family of *regular polynomial endomorphisms of the affine space* $\mathbb{A}^k$, see [BJ]. Choosing an affine chart, we can assume the hyperplane H_∞ is the hyperplane at infinity of $\mathbb{A}^k$ in $\mathbb{P}^k$. When $(\mathbb{P}^k \times S, f, \mathcal{O}_{\mathbb{P}^k}(1))$ is such a family of regular polynomial endomorphisms, we let

$$G_{f_t}(z) = G_f(z,t) := \lim_{n\to\infty} \frac{1}{d^n} \log^+ \|f_t^n(z)\|,$$

for all $z \in \mathbb{C}^k$ and all $t \in S^0(\mathbb{C})$.

We let $\mathcal{Y} \subset \mathbb{P}^k \times S$ be an irreducible hypersurface that projects surjectively onto S and which intersects properly $H_\infty \times S$. Up to reducing the Zariski open set S^0, we can assume $\mathcal{Y}$ is flat over S^0 and $Y_t \neq H_\infty$ for all $t \in S^0$.

Definition 7.1. *The* polynomial bifurcation measure *of the pair* $(\mathbb{P}^k \times S, f, \mathcal{O}_{\mathbb{P}^k}(1), \mathcal{Y})$ *is the Monge-Ampère measure associated to the function* $G_{f,\mathcal{Y}} : S^0(\mathbb{C}) \to \mathbb{R}^+$ *defined by*

$$G_{f,\mathcal{Y}}(t) := \int_{\mathbb{C}^k} G_{f_t}(dd^c G_{f_t}(z))^{k-1} \wedge [Y_t], \quad t \in S^0(\mathbb{C}).$$

We denote by $\mu_{f\mathcal{Y}}^{\mathrm{pol}} := (dd^c G_{f,\mathcal{Y}})^{\dim S}$ *this measure.*

The measure $\mu_{f\mathcal{Y}}^{\mathrm{pol}}$ is perfectly adapted to detect phenomena which occur in the affine space. However, it does not in general allow to collect all the informations that $\mu_{f,\mathcal{Y}}$ carries. However, as measures on S^0, we clearly have

$$\mu_{f,\mathcal{Y}} \geq \mu_{f,\mathcal{Y}}^{\mathrm{pol}}.$$

We now prove here the following which is a sufficient condition to get a height gap, and then to deduce uniformity in a Bogomolov-type statement.

Theorem 7.2. *Let* $(\mathbb{P}^k \times S, f, \mathcal{O}_{\mathbb{P}^k}(1))$ *be a family of regular polynomial endomorphisms of degree* d *of the affine space parametrized by* S*, all defined over a number field and let* $\mathcal{Y} \subset \mathbb{P}^k \times S$ *be an irreducible hypersurface such that* $\pi|_{\mathcal{Y}} : \mathcal{Y} \to S$ *is surjective which intersects properly* $H_\infty \times S$*. Assume there is a non-empty analytic open subset* $\Omega \subset S^0(\mathbb{C})$ *that is contained in* $\mathrm{supp}(\mu_{f,\mathcal{Y}}^{\mathrm{pol}})$*.*

Then there exists $Z \subsetneq S$ Zariski-closed, $\varepsilon > 0$ and an integer $N \geq 1$ such that for all $t \in (S^0 \setminus Z)(\bar{\mathbb{Q}})$, there exists $W_t \subsetneq Y_t$ with $\deg(W_t) \leq N$ and such that

$$\{z \in Y_t(\bar{\mathbb{Q}}) \,:\, \widehat{h}_{f_t}(z) \leq \varepsilon\} \subset W_t.$$

Remark 7.3. By Zhang's inequalities (12), this in particular implies that

$$\widehat{h}_{f_t}(Y_t) \geq \frac{\varepsilon}{k} > 0, \quad \text{for all} \quad t \in (S^0 \setminus Z)(\bar{\mathbb{Q}}).$$

For our proof, we rely on the next lemma, which is of purely complex analytic nature.

Lemma 7.4. *Let $(\mathbb{P}^k \times M, f, \mathcal{O}_{\mathbb{P}^k}(1))$ be a complex family of regular polynomial endomorphisms of the affine space of degree d parametrized by S of dimension m. Assume there exists a non-empty open subset $\Omega \subset \mathrm{supp}(\mu^{\mathrm{pol}}_{f,\mathcal{Y}})$. Then*

$$\int_{\mathbb{C}^{k(m+1)} \times S} G_{f^{[m+1]}} (dd^c G_{f^{[m+1]}})^{km+k-1} \wedge [\mathcal{Y}^{[m+1]}] > 0.$$

Proof. We denote by $p_i : (\mathbb{P}^k)^{m+1} \times S \to \mathbb{P}^k \times S$ the projection onto the i-th factor of the fiber product and by $\pi_i : (\mathbb{P}^k)^{m+1} \times S \to (\mathbb{P}^k)^m \times S$ the projection consisting in forgetting the i-th factor. By construction $f^{[m]}$ and $f^{[m+1]}$ are families of regular polynomial endomorphisms of the affine spaces $\mathbb{A}^{km}$ and $\mathbb{A}^{k(m+1)}$ respectively. Moreover, for any $1 \leq i \leq m+1$, we have

$$G_{f^{[m+1]}} = \sum_{j=1}^{m+1} G_f \circ p_j = G_f \circ p_i + G_{f^{[m]}} \circ \pi_i.$$

Using that $[\mathcal{Y}^{[m+1]}] = p_1^*[\mathcal{Y}] \wedge \pi_1^*[\mathcal{Y}^{[m]}]$, we find

$$\begin{aligned}
I :=& \int_{\mathbb{C}^{k(m+1)} \times S} G_{f^{[m+1]}} (dd^c G_{f^{[m+1]}})^{km+k-1} \wedge [\mathcal{Y}^{[m+1]}] \\
\geq& \int_{\mathbb{C}^{k(m+1)} \times S} G_f \circ p_1 (p_1^* dd^c G_f + \pi_1^* dd^c G_{f^{[m]}})^{km+k-1} \wedge [\mathcal{Y}^{[m+1]}] \\
\geq& \int_{\mathbb{C}^{k(m+1)} \times S} G_f \circ p_1 (p_1^* dd^c G_f)^{k-1} \wedge (\pi_1^* dd^c G_{f^{[m]}})^{km} \wedge [\mathcal{Y}^{[m+1]}] \\
=& \int_{\mathbb{C}^{k(m+1)} \times S} p_1^* \left(G_f (dd^c G_f)^{k-1} \wedge [\mathcal{Y}] \right) \wedge \pi_1^* \left((dd^c G_{f^{[m]}})^{km} \wedge [\mathcal{Y}^{[m]}] \right) \\
=& \int_{\mathbb{C}^{km} \times S} \left(\int_{\mathbb{C}^k} G_{f_t} (dd^c G_{f_t})^{k-1} \wedge [Y_t] \right) (dd^c G_{f^{[m]}})^{km} \wedge [\mathcal{Y}^{[m]}] \\
=& \int_{\mathbb{C}^{km} \times S} G_{f,\mathcal{Y}} \circ \pi_{[m]} \cdot (dd^c G_{f^{[m]}})^{km} \wedge [\mathcal{Y}^{[m]}].
\end{aligned}$$

Claim. *For $m \geq \dim S$, there is $C(m) \geq 1$ such that*

$$(\pi_{[m]})_* \left((dd^c G_{f^{[m]}})^{km} \wedge [\mathcal{Y}^{[m]}] \right) = C(m) \mu^{\mathrm{pol}}_{f,\mathcal{Y}}.$$

According to the Claim above, we find

$$I \geq \int_S G_{f,\mathcal{Y}} \cdot \mu^{\mathrm{pol}}_{f,\mathcal{Y}}.$$

We now proceed by contradiction. If the last integral vanishes, for any $\varepsilon > 0$, the set of points $t \in \Omega$ such that $G_{f,\mathcal{Y}}(t) \leq \varepsilon$ is dense in Ω. As $G_{f,\mathcal{Y}} \geq 0$, this implies the continuous

function $G_{f,\mathcal{Y}} : S^0(\mathbb{C}) \to \mathbb{R}$ is constant equal to zero on Ω. This is a contradiction since $(dd^c G_{f,\mathcal{Y}})^m$ would be zero on Ω. □

All there is left to do is to prove the Claim.

Proof of the Claim. We first prove that $dd^c G_{f,\mathcal{Y}} = \pi_* \left((dd^c G_f)^k \wedge [\mathcal{Y}]\right)$ using a slicing argument. Indeed, if ϕ is a smooth compactly supported $(\dim S - 1, \dim S - 1)$-form on $S^0(\mathbb{C})$, we have

$$\begin{aligned}\int_{\mathcal{X}^0(\mathbb{C})} \pi^*\phi \wedge (dd^c G_f)^k \wedge [\mathcal{Y}] &= \int_{\mathcal{X}^0(\mathbb{C})} G_f (dd^c G_f)^{k-1} \wedge [\mathcal{Y}] \wedge \pi^*(dd^c\phi) \\ &= \int_{S^0(\mathbb{C})} \left(\int_{\pi^{-1}\{t\}} G_{f_t} (dd^c G_{f_t})^{k-1} \wedge (\iota_t)^*[\mathcal{Y}] \right) \cdot dd^c\phi,\end{aligned}$$

where $\iota_t : X_t \to \mathcal{X}$ is the natural injection, so that $\iota_t^*[\mathcal{Y}] = [Y_t]$, whence

$$\int_{\mathcal{X}^0(\mathbb{C})} \pi^*\phi \cdot (dd^c G_f)^k \wedge [\mathcal{Y}] = \int_{S^0(\mathbb{C})} G_{f,\mathcal{Y}} \cdot dd^c\phi = \int_{S^0(\mathbb{C})} dd^c G_{f,\mathcal{Y}} \wedge \phi.$$

To conclude, we proceed as in the proof of Proposition 1.4. □

Now, when X is a projective variety and L is a line bundle on X, we denote by $L^{\boxtimes N}$ the induces line bundle on X^N, i.e. $L^{\boxtimes N} = \tau_1^* L + \cdots + \tau_N^* L$, where $\tau_i : X^N \to X$ is the canonical projection onto the i-th coordinate. We will also use the next Lemma due to Gao, Ge and Kühne [GGK, Lemma 4.3].

Lemma 7.5. *Let X be an irreducible projective variety with a very ample line bundle L, defined over an algebraically closed field K and $N \geq 2$. Let $Z \subsetneq X^N$ be a proper closed subvariety. There exists a constant*

$$B = B(N, \dim X, \deg_L(X), \deg_{L^{\boxtimes N}}(Z)) > 0,$$

such that for any subset $\Sigma \subset X(K)$ with $\Sigma^N \subseteq Z(K)$, there exists a proper closed subvariety X' of X with $\Sigma \subset X'(K)$ and $\deg_L(X') < B$.

We are now in position to prove Theorem 7.2.

Proof of Theorem 7.2. For any $v \in M_{\mathbb{K}}$, recall that the Green function of $f^{[m+1]}$ is

$$G_{f^{[m+1]},v}(x) := \lim_{n\to\infty} \sum_{j=1}^{m+1} \frac{1}{d^n} \log^+ \|f^n \circ p_i(x)\|_v, \quad x \in \mathbb{A}^{k(m+1)}(\bar{\mathbb{Q}}) \times S^0(\bar{\mathbb{Q}}).$$

One can easily show that for any $x \in \mathbb{A}^{k(m+1)}(\bar{\mathbb{Q}}) \times S^0(\bar{\mathbb{Q}})$, we have

$$\widehat{h}_{f^{[m+1]}}(x) = \frac{1}{[\mathbb{L} : \mathbb{K}]} \sum_{v \in M_{\mathbb{K}}} \sum_{\sigma \in \mathrm{Gal}(\mathbb{L}/\mathbb{K})} n_v G_{f^{[m+1]},v}(\sigma(x)),$$

where $\mathbb{L}$ is any finite extension of $\mathbb{K}$ so that $x \in \mathbb{A}^{k(m+1)}(\mathbb{L}) \times S^0(\mathbb{L})$. In particular, for a given place $v \in M_{\mathbb{K}}$, we deduce that

$$\frac{n_v}{[\mathbb{L} : \mathbb{K}]} \sum_{\sigma \in \mathrm{Gal}(\mathbb{L}/\mathbb{K})} G_{f^{[m+1]},v}(\sigma(x)) \leq \widehat{h}_{f^{[m+1]}}(x). \tag{16}$$

We proceed by contradiction, assuming that there is a Zariski dense subset of small points, i.e. for all $\varepsilon > 0$, the set

$$E_\varepsilon := \{x \in \mathcal{Y}^{[m+1]}(\bar{\mathbb{Q}}) \,:\, \widehat{h}_{f^{[m+1]}}(x) \leq \varepsilon\}$$

is Zariski dense in $\mathcal{Y}^{[m+1]}(\bar{\mathbb{Q}})$. In particular, there exists a generic sequence $(x_n) \in (\mathcal{Y}^{[m+1]})^0(\bar{\mathbb{Q}})$ such that $\widehat{h}_{f^{[m+1]}}(x_n) \to 0$ as $n \to \infty$. Let now $v_0 \in M_{\mathbb{K}}$ be an archimedean place. Since we will now work only at the place v_0, we forget the subscript v_0 in the rest of the proof.

By construction of the Green current $\widehat{T}_{f^{[m+1]}}$ and of the Green function $G := G_{f^{[m+1]}}$, as measures on $\mathbb{C}^{k(m+1)} \times S^0(\mathbb{C})$, we have

$$\widehat{T}_{f^{[m+1]}}^{k(m+1)-1} \wedge [\mathcal{Y}^{[m+1]}] = (dd^c G_{f^{[m+1]}})^{k(m+1)-1} \wedge [\mathcal{Y}^{[m+1]}].$$

In particular, Lemma 7.4 says that

$$\int_{(\mathcal{Y}^{[m+1]})^0_\star(\mathbb{C})} G \cdot \mu_{m+1} > 0,$$

where $\mu_{m+1} = \widehat{T}_{f^{[m+1]}}^{\dim \mathcal{Y}^{[m+1]}}$ and $(\mathcal{Y}^{[m+1]})^0_\star = (\mathcal{Y}^{[m+1]})^0 \cap (\mathbb{A}^{k(m+1)} \times S)$.

As G is continuous and non-negative on $(\mathcal{Y}^{[m+1]})^0_\star(\mathbb{C})$, we deduce that there exists a non-empty open analytic set $\Omega \Subset (\mathcal{Y}^{[m+1]})^0_\star(\mathbb{C})$ such that $G > 0$ on Ω and such that $\mu_m(\Omega) > 0$. Let $\chi : (\mathcal{Y}^{[m+1]})^0_\star(\mathbb{C}) \to \mathbb{R}_+$ be a smooth compactly supported function with $\chi = 1$ on Ω and $0 \leq \chi \leq 1$. The function $\phi := G \cdot \chi$ is thus continuous, compactly supported, and $G \geq \phi$. We now apply the Equidistribution Theorem 6.1:

$$\lim_{n\to\infty} \frac{1}{\mathrm{Card}(\mathsf{O}(x_n))} \sum_{y \in \mathsf{O}(x_n)} \phi(y) = \int_{(\mathcal{Y}^{[m+1]})^0_\star(\mathbb{C})} \phi \mu_{m+1}.$$

In particular, there is $n_0 \geq 1$ such that for any $n \geq n_0$, we have

$$\frac{1}{\mathrm{Card}(\mathsf{O}(x_n))} \sum_{y \in \mathsf{O}(x_n)} \phi(y) \geq \frac{1}{2} \int_{(\mathcal{Y}^{[m+1]})^0_\star(\mathbb{C})} \phi \mu_{m+1} \geq \frac{1}{2} \int_\Omega G \mu_{m+1} > 0.$$

Moreover, for any finite extension $\mathbb{L}_n$ of $\mathbb{K}$ with $x_n \in \mathcal{Y}^{[m+1]}(\mathbb{L}_n)$,

$$\begin{aligned} \frac{1}{[\mathbb{L}_n : \mathbb{K}]} \sum_{\sigma \in \mathrm{Gal}(\mathbb{L}_n/\mathbb{K})} G_{f^{[m+1]}}(\sigma(x_n)) &= \frac{1}{\mathrm{Card}(\mathsf{O}(x_n))} \sum_{y \in \mathsf{O}(x_n)} G(y) \\ &\geq \frac{1}{\mathrm{Card}(\mathsf{O}(x_n))} \sum_{y \in \mathsf{O}(x_n)} \phi(y), \end{aligned}$$

where we used that $G \geq \phi$. Together with (16), this gives

$$\widehat{h}_{f^{[m+1]}}(x_n) \geq \frac{n_{v_0}}{2} \int_\Omega G \mu_{m+1} > 0,$$

for any $n \geq n_0$. This is a contradiction since $\widehat{h}_{f^{[m+1]}}(x_n) \to 0$ as $n \to \infty$ by hypothesis.

We have thus proved there exists $\varepsilon_0 > 0$ such that the set E_{ε_0} is not Zariski dense in $\mathcal{Y}^{[m+1]}(\bar{\mathbb{Q}})$. In particular, there is a proper Zariski closed subset $V \subsetneq \mathcal{Y}^{[m+1]}$ which is defined over $\bar{\mathbb{Q}}$ and that contains E_{ε_0}. If $Z := \pi_{[m+1]}(V) \subsetneq S$ is a proper closed subvariety of S, then for any $t \in (S^0 \setminus Z)(\bar{\mathbb{Q}})$, we have $\widehat{h}_{f^{[m+1]}} \geq \varepsilon_0$ on $Y_t^{m+1}(\bar{\mathbb{Q}})$. It is in particular true on $\Delta := \{(z, \dots, z) : z \in Y_t(\bar{\mathbb{Q}})\}$. Let $\varepsilon := \varepsilon_0/(m+1)$. This gives

$$\sum_{j=1}^{m+1} \widehat{h}_{f_t}(z) = \widehat{h}_{f^{[m+1]}}(z, \dots, z, t) \geq \varepsilon_0,$$

which rewrites as $\widehat{h}_{f_t} \geq \varepsilon = \varepsilon_0/(m+1)$ on $Y_t(\bar{\mathbb{Q}})$.

Assume now that $\pi(V) = S$ and let $Z \subsetneq S$ be the proper closed subvariety of S such that $\pi_{[m+1]}$ is flat on each irreducible component of V over $S^0 \setminus Z$. Pick now $t \in (S^0 \setminus Z)(\bar{\mathbb{Q}})$. By definition, the line bundle $L_t := \mathcal{O}_{\mathbb{P}^k}(1)|_{Y_t}$ is very ample and the set $V_t := V \cap Y_t$ is a proper closed subvariety of Y_t with $D := \deg_{L_t}(V_t)$ independent of t. Let

$$\Sigma_t := \left\{ z \in Y_t(\bar{\mathbb{Q}}) \,:\, \widehat{h}_{f_t}(z) \leq \varepsilon \right\}.$$

where $\varepsilon = \varepsilon_0/(m+1)$ as above. The conclusion follows from Lemma 7.5. □

7.3. **Uniformity in the moduli space $\mathscr{P}_d^2$.** We now focus on the universal family

$$f : \mathbb{P}^2_{\mathscr{P}_d^2} \longrightarrow \mathbb{P}^2_{\mathscr{P}_d^2}$$

of the moduli space $\mathscr{P}_d^2$ and as above, this is a family $(\mathbb{P}^2_S, f, \mathcal{O}_{\mathbb{P}^2}(1))$ of degree d endomorphisms of $\mathbb{P}^2$ parametrized by a projective model S of $\mathscr{P}_d^2$ with regular part $\mathscr{P}_d^2$ which is defined over $\mathbb{Q}$ (see § 2.1).

We also study here the variation of the canonical height of the critical locus. However, when $f : \mathbb{A}^2 \to \mathbb{A}^2$ is a degree d regular polynomial endomorphism, L_∞ is an irreducible component of the critical locus of f and f induces an endomorphism of L_∞, we denote by f_{L_∞}. This induces a map

$$r : \mathscr{P}_d^2 \longrightarrow \mathscr{M}_d^1$$

defined by $r(t) = [f_{t,L_\infty}]$. This map is well defined and surjective and, for every $[g] \in \mathscr{M}_d^1$, the set $r^{-1}([g])$ consists of conjugacy classes of regular polynomial automorphisms whose restriction to L_∞ are conjugated to g. It thus is a subvariety of $\mathscr{P}_d^2$ of dimension $\mathcal{P}_d^2 - \mathcal{N}_d^1 > 0$.

In the present situation, one easily sees that $\mathrm{Crit}(f_t)$ decomposes as

$$\mathrm{Crit}(f_t) = L_\infty \cup C_{f_t}$$

where $C_{f_t} \cap \mathbb{A}^2 = \{z \in \mathbb{A}^2 \,:\, \det(D_z f_t) = 0\} = \mathrm{Crit}(f_t) \cap \mathbb{A}^2$. We now let

$$\mathcal{C} := \{(z,t) \in \mathbb{P}^2 \times \mathscr{P}_d^2 \,:\, z \in C_{f_t}\}.$$

The next key lemma is a consequence of Theorem C (see Theorem 4.1).

Lemma 7.6. *There exists a non-empty open set $\Omega \subset \mathscr{P}_d^2(\mathbb{C})$ that is contained in* $\mathrm{supp}(\mu_{f,\mathcal{C}}^{\mathrm{poly}})$.

Proof. Write $P := \mathcal{P}_d^2$ for simplicity. First, as currents on $\mathbb{C}^{2P} \times \mathscr{P}_d^2(\mathbb{C})$, we have

$$\widehat{T}_{f^{[P]}} = dd^c G_{f^{[P]}}.$$

In particular, as measures on $\mathbb{C}^{2P} \times \mathscr{P}_d^2(\mathbb{C})$, we also have

$$\widehat{T}^{2P}_{f^{[P]}} \wedge [\mathrm{Crit}^{[P]}] = \widehat{T}^{2P}_{f^{[P]}} \wedge [\mathcal{C}^{[P]}] = \left(dd^c G_{f^{[P]}}\right)^{2P} \wedge [\mathcal{C}^{[P]}].$$

By construction, the points of the support of $\widehat{T}^{2P}_{f^{[P]}} \wedge [\mathrm{Crit}^{[P]}]$ constructed in Theorem 4.1 belong to $\mathbb{C}^{2P} \times \mathscr{P}_d^2(\mathbb{C})$. In particular, they belong to the support of $\left(dd^c G_{f^{[P]}}\right)^{2P} \wedge [\mathcal{C}^{[P]}]$. We conclude by pushing forward this measure by $\pi_{[P]}$. □

We are now in position to prove the following result.

Theorem 7.7. *Fix $d \geq 2$. There are constants $B(d) \geq 1$ and $\varepsilon(d) > 0$ and a non-empty Zariski open subset $U \subset \mathrm{Poly}_d^2$ such for any $f \in U(\bar{\mathbb{Q}})$, then*

$$\#\{z \in C_f(\bar{\mathbb{Q}}) \,:\, \widehat{h}_f(z) \leq \varepsilon(d)\} \leq B(d).$$

Proof. As before, we let

$$f : \mathbb{P}^2_{\mathscr{P}^2_d} \to \mathbb{P}^2_{\mathscr{P}^2_d}$$

be the universal family. Lemma 7.6 with Theorem 7.2 imply that there are $\varepsilon > 0$, $B \geq 1$ and a non-empty Zariski open set $\mathscr{U} \subset \mathscr{P}^2_d$ such that for any $t \in \mathscr{U}(\bar{\mathbb{Q}})$

$$\#\{z \in C_{f_t}(\bar{\mathbb{Q}}) \,:\, \widehat{h}_{f_t}(z) \leq \varepsilon\} \leq B.$$

Up to reducing $\mathscr{U}$, we can assume $\mathscr{U} \cap \Pi(V) = \varnothing$, where V is such that $\Pi : \mathrm{Poly}^2_d \setminus V \to \mathscr{P}^2_d \setminus \Pi(V)$ is a principal bundle. The set $U := \Pi^{-1}(\mathscr{U})$ is Zariski open and, for any $f \in U(\bar{\mathbb{Q}})$, if $t = \Pi(f)$, there is $\phi \in \mathrm{Aut}_{\bar{\mathbb{Q}}}(\mathbb{A}^2)$ such that $\phi \circ f = f_t \circ \phi$ on $\mathbb{P}^2$. Let $\bar{L}_f$ (resp. $\bar{L}_{f_t}$) be the line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ endowed with the semi-positive continuous adelic f-invariant (resp. f_t-invariant) metric. Fix a number field $\mathbb{L}$ such that f, f_t and ϕ are all defined over $\mathbb{L}$ and let $v \in M_{\mathbb{L}}$. If $\|\cdot\|_{f,v}$ and $\|\cdot\|_{f_t,v}$ are the metrics invariant by f and f_t respectively at the place v, the equation $\phi \circ f = f_t \circ \phi$ easily gives

$$\phi^* \|\cdot\|_{f,v} = \|\cdot\|_{f_t,v} \quad \text{on} \quad L_v.$$

In particular, for any closed subvariety $Y \subset \mathbb{P}^2$ which is defined over $\bar{\mathbb{Q}}$, we deduce

$$\widehat{h}_f(\phi^{-1}(Y)) = \widehat{h}_{f_t}(Y).$$

As $\phi \circ f = f_t \circ \phi$, we find $C_f = \phi^{-1}(C_{f_t})$ so that $\widehat{h}_f(C_f) = \widehat{h}_{f_t}(C_{f_t})$ and

$$\{z \in C_f(\bar{\mathbb{Q}}) \,:\, \widehat{h}_f(z) \leq \varepsilon\} = \phi^{-1}\left(\{z \in C_{f_t}(\bar{\mathbb{Q}}) \,:\, \widehat{h}_{f_t}(z) \leq \varepsilon\}\right).$$

As ϕ is an automorphism of $\mathbb{P}^2$, the conclusion follows. □

To conclude, it remains to prove Theorem D.

Proof of Theorem D . Observe that the statement is a direct consequence of Theorem D in $\bar{\mathbb{Q}}$: there exists a constant $B(d) \geq 1$ and a non-empty Zariski open subset $U \subset \mathrm{Poly}^2_d$ such for any $f \in U(\bar{\mathbb{Q}})$, we have

$$\#\mathrm{Preper}(f) \cap C_f \leq B(d).$$

Now, let $f \in U(\mathbb{C})$ with $\#\mathrm{Preper}(f) \cap C_f \geq B(d) + 1$. The equations defining these $B(d) + 1$ points are defined over $\bar{\mathbb{Q}}$ (we only consider the periods and preperiods attached to those precised points) so we reach a contradiction. □

7.4. **The case of polynomial skew-products.** A regular polynomial endomorphism $f : \mathbb{A}^2 \to \mathbb{A}^2$ of degree d defined over a field K is a *skew-product* if we have

$$f(z, w) = (p(z), q(z, w)),$$

where $p \in K[z]$ has degree d and $q(z, w) \in K[z, w]$ satisfies $\deg_w(q) = d$. Following Astorg and Bianchi [AB], we denote by $\mathrm{Sk}(p, d)$ the space of all degree d polynomial skew-products with given base p, up to affine conjugacy. It is not difficult to see that, when p is defined over a number field $\mathbb{K}$, then $\mathrm{Sk}(p, d)$ is also defined over $\mathbb{K}$.

As in the previous case, this space is a fine moduli space whence there is a universal family $f : \mathbb{P}^2_{\mathrm{Sk}(p,d)} \longrightarrow \mathbb{P}^2_{\mathrm{Sk}(p,d)}$ (note that it can be made explicit in this case, see [AB]). Astorg and Bianchi show the following.

Theorem 7.8 ([AB])**.** *Let $p \in \mathbb{C}[z]$ be a polynomial whose Julia set is not totally disconnected and which is not conjugated to z^d or to the degree d Chebyshev polynomial. Then the support of $\mu_{f,\mathrm{Crit}}$ has non-empty interior in* $\mathrm{Sk}(p, d)(\mathbb{C})$.

Note that all postcritically finite polynomials have connected Julia set and are defined over $\bar{\mathbb{Q}}$. Note also that a polynomial skew-product $f = (p, q)$ can be PCF only if its base p is itself PCF. As a direct application of Theorem 7.2 and of Theorem 7.8, as above we have the following.

Theorem 7.9. *Let $p \in \mathbb{C}[z]$ be a PCF polynomial which is not conjugated to z^d or to the degree d Chebyshev polynomial. Then there exists a non-empty Zariski open subset $U \subset \mathrm{Sk}(p, d)$, and constants $\varepsilon(p, d) > 0$ and $B(p, d) \geq 1$ such that for any $f \in U(\bar{\mathbb{Q}})$, then*

$$\#\{z \in C_f(\bar{\mathbb{Q}}) \,:\, \widehat{h}_f(z) \leq \varepsilon(p, d)\} \leq B(p, d).$$

In particular, for any $f \in U(\mathbb{C})$, we have

$$\#\mathrm{Preper}(f) \cap C_f(\mathbb{C}) \leq B(p, d).$$

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