Université de Bourgogne-Franche-Comté

École Doctorale Carnot–Pasteur

Habilitation à Diriger des Recherches

Discipline : Mathématiques

présentée par

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Attracteurs et Bifurcations en Dynamique Holomorphe

Soutenue le 20 novembre 2019 devant le jury composé de :

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Remerciements

Constituer mon jury d'HDR a été une bonne occasion de réunir des personnes qui ont marqué mon parcours depuis ma thèse. En premier lieu, il y a Christian Bonatti qui m'a ouvert les yeux sur bien des aspects de la dynamique et m'a rapidement fait comprendre que, malgré mes réticences initiales, Dijon était l'un des meilleurs lieux où je pouvais poser mes valises mathématiques. Bien que mes outils lui sont en grande partie étrangers, il était pour moi un choix naturel de rapporteur et je le remercie d'avoir accepté cette tâche.

A l'opposé, Romain Dujardin est sans doute l'une des personnes les plus familières avec mes problématiques et mes techniques et je le remercie d'avoir apporté son expertise à ce manuscrit. Cette proximité l'a aussi amené à devancer certains de mes travaux mais je le remercie d'avoir été plus modeste que Fermat dans ses énoncés de conjectures.

J'exprime aussi ma gratitude aux autres membres du jury : à Nessim Sibony pour avoir été l'un des pionniers de ce beau et riche sujet qu'est la dynamique complexe, à Henry de Thélin pour ses résultats nets et fort utiles, à Pierre Berger pour secouer un peu le monde mathématique autour de lui et à Pavao Mardesic pour ses marques d'intérêt et de gentillesse.

Il y a aussi ceux qui n'ont pas pu venir, et je remercie donc évidement Misha Lyubich pour avoir rapporté ce texte, mais aussi François Berteloot pour sa bienveillance, pour avoir initié une théorie qui va sans doute m'accompagner un petit temps et pour ses qualités de directeur de thèses qui m'ont déjà donné deux coauteurs !

D'ailleurs, je remercie Sandrine, Fabrizio et Christophe pour avoir partagé leurs idées et leur bonne humeur durant nos collaborations. Je remercie également tous les membres et participants de l'ANR Fatou pour l'excellente ambiance qui règne dans ce groupe et en particulier Thomas et Gabriel pour être à l'origine de ce projet.

Durant toutes ces années l'IMB fut un excellent lieu de vie et de travail. Je remercie tous ceux qui m'ont accueilli (Luis, Adrien, Guiseppe, Ahmed, Olivier, Lucy et tant d'autres), ceux qui nous ont rejoint pour rajeunir nos rang (comme Sébastien ou la team des anciens Lyonnais (dont Marielle!!)), ceux qui prennent bien trop soin de moi (en particulier Marie-Claire mais Jose-Luis n'est pas loin) et même ceux qui ont préféré nous quitter (n'est-ce pas Emmanuel, Sébastien ou Caroline ? et je ne parle pas des anciens doctorants !).

Enfin, mes derniers remerciements vont à Laure et Raphaël, mes précieux antidotes au quotidien contre toute forme de morosité.

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Introduction

In discrete-time complex dynamics, we consider a holomorphic self-map f of a complex manifold and we study the behavior of the iterates $(f^n)_{n\geq 1}$. Holomorphic maps are much more rigid objects than continuous or smooth ones so it is therefore natural to ask the following two questions:

- What are the constraints on the system induced by the holomorphic condition?
- How much freedoms does this condition leave?

Although very similar, these two questions lead to different approaches. One goal, compared to the former, is to obtain global results that distinguish holomorphic systems from smooth ones. For the latter, one is rather led to look for examples that exhibit interesting phenomena. In some sense, these two questions bound, respectively from above and from below, the complexity that a holomorphic dynamical system can have.

Here are two simple examples.

(1) If $f\colon U \to \mathbb{C}^k$ is a holomorphic map defined on a bounded open subset U of $\mathbb{C}^k$ such that $\overline{f(U)} \subset U$ then $A := \cap_{n\geq 0} f^n(U)$ is a finite union of attracting cycles and all the orbits of f accumulate on A. This ensures that several local phenomena which appear in smooth dynamics (during the unfolding of a homoclinic tangency for example) cannot exist in complex dynamics.

(2) On the other hand, if $\overline{U} \subset f(U)$ then the dynamics of f can be rich. Actually, if f is a *polynomial-like map* (i.e. $f(U)$ is convex and $f\colon U \to f(U)$ is proper) which is not invertible then Dinh-Sibony proved in [DS03] that f possesses a natural invariant measure μ, called the *equilibrium measure*, which is mixing, of maximal (and positive) entropy. In particular, μ has one positive Lyapunov exponent. Nevertheless, it is still an open problem whether all the Lyapunov exponent of μ have to be positive or, more simply, if f has to possess a repelling periodic point.

Under a technical assumption on f ("of large/dominant topological degree") Dinh and Sibony answer positively to both questions in **(2)** using several techniques, now called the "dd^c-method", which heavily rely on pluripotential theory.

My main area of research concerns the dynamics of holomorphic endomorphisms of the complex projective space $\mathbb{P}^k$ of dimension $k \geq 2$. This manifold has several advantages which are not common in the complex world:

- it is compact,
- the dynamics of an endomorphism of $\mathbb{P}^k$ is far from being trivial in general,
- its set of endomorphisms is sufficiently large to provide interesting bifurcation phenomena.

In order to the study such an endomorphism f, one can hope (perhaps naively) that the space $\mathbb{P}^k$ splits into countably many pieces such that

- we have some information on the dynamics on the different pieces,
- the bifurcations of f (i.e. the potential drastic changes in the dynamics when f is perturbed) come from specific phenomena in some piece that we can understand.

In dimension one, when $k = 1$, such a decomposition is given by the dichotomy between the Fatou set and the Julia set. In higher dimension, this Fatou/Julia dichotomy still exists but the dynamics on these sets is far from being understood, especially on the Julia set. One can instead consider the partition of $\mathbb{P}^k$ into chain recurrence classes and a non-recurrent set (see Section 1.2 for definitions). The map f always possesses a canonical "maximal" chain recurrence class. In fact, like in **(2)**, it is possible to construct an equilibrium measure μ associated to f with several nice properties. Its support, often called the "small Julia set", plays a central role and it catches the most chaotic and most repulsive part of the dynamics. As the measure μ is ergodic, this small Julia set corresponds to a chain recurrence class but little is known in general about the dynamics outside this class. One way to study this dynamics (see Section 1.2 for more details) is to consider an open subset $U \subsetneq \mathbb{P}^k$ such that $\overline{f(U)} \subset U$. Although similar to **(1)**, it turns out that the situation in $\mathbb{P}^k$ is much less trivial than in $\mathbb{C}^k$ and the dynamics on the set $A := \cap_{n\geq 0} f(U)$ can be very rich. However, in [Taf18] I obtain significative restrictions on the geometry of A and on the dynamics of $f_{|A}$. The starting point of this study is to show that, although there is no additional assumption like in **(2)**, the simple inclusion $\overline{f(U)} \subset U$ in $\mathbb{P}^k$ puts geometric contraints on U which allow to adapt the dd^c-method of Dinh-Sibony. An illustration of the results of [Taf18] is the following theorem where the dynamics on A is assumed to be topologically transitive (i.e. when A is an "attractor").

Theorem 0.0.1. *Let f and A be as above. If $f_{|A}$ is topologically transitive then there exist two integers $0 \leq s \leq k$ and $n_0 \geq 1$ such that*

- *the topological entropy of $f_{|A}$ is $s \log d$,*
- *A is the disjoint union of connected compact sets $A_1, \dots, A_{n_0}$ invariant by f^{n_0},*
- *on each A_i, there exists a measures ν_i which is mixing for f^{n_0}, of maximal entropy $s\log(d^{n_0})$ on A_i and with at least s positive Lyapunov exponents.*

As a trivial example, when A is an attractive cycle then $s = 0$ and n_0 is the period of the cycle.

In general, in the setting of Theorem 0.0.1 the decomposition $A = A_1 \cup \dots \cup A_{n_0}$ into finitely connected components is easy to obtain and it already holds for continuous maps. However, in [Taf18] I consider a more general type of objects, called quasi-attractors, where this decomposition still holds on $\mathbb{P}^k$ and is a non-trivial fact. The advantage of this generalization is that some chain recurrence classes (the "minimal" ones) are quasi-attractors hence the counterpart to Theorem 0.0.1 in that setting puts strong contraints on these classes which are clearly not satisfy in smooth dynamics. Notice also that the existence of mixing measures (of maximal entropy or not) for some power of f is not true for smooth maps (as irrational rotations of the circle).

Like in **(2)**, it would be very satisfying to show that the inclusion $\overline{f(U)} \subset U$ implies the existence of a hyperbolic measure, or more simply, of a hyperbolic periodic point in U. Unfortunately, I was not able to prove that the measures ν_i in Theorem 0.0.1 are always hyperbolic (which is very likely). However, in [Taf13] and [DT18a], we put additional assumptions on the open set U and on the map f (which are more or less satisfied in all the known examples) which imply in particular that the resulting measures are hyperbolic. Moreover, the framework of [DT18a] (originally introduced by Daurat in [Dau14]) is interesting by itself and we show that it is satisfied by a large class of endomorphisms of $\mathbb{P}^k$ (see also [Dau14] when $k = 2$). In [Taf17], I also provide examples, increasing slightly the range of the possible attractors in $\mathbb{P}^k$. All this is developed in Chapter 1.

Chapter 2 is about bifurcation theory for endomorphisms of $\mathbb{P}^k$. As mentioned above, the small Julia set supports the most chaotic part of the dynamics and thus it is natural, at first, to study bifurcations with respect to this set. Such a theory has been recently developed by Berteloot-Bianchi-Dupont on $\mathbb{P}^k$ for $k \geq 2$ and my motivation in this subject is to exhibit phenomena which distinguish this theory from the more classical theory of Mañé-Sad-Sullivan and Lyubich when $k = 1$. In dimension 1, structural stability is always dense in the parameter space and almost all the different possible types of bifurcations coincide. In particular, a bifurcation implies that the Julia set does not depend continuously on the parameter. In [BT17], with Bianchi we study a very specific family of endomorphisms of $\mathbb{P}^2$ where there is an open set of bifurcations and the Julia set depends continuously on the parameter in the whole family. And my main contribution regarding bifurcation is the use of objects coming from smooth dynamics called *blenders*. In fact, Dujardin introduced them in complex dynamics in order to obtain open sets of bifurcations in the family of all endomorphisms of $\mathbb{P}^k$, $k \geq 2$. In [Taf17], I was able to strengthen his result, proving that blenders always appear near bifurcations of product maps of $\mathbb{C}^2$.

All the examples above, like many others in the recent literature, start from an endomorphism of $\mathbb{P}^2$ which preserves a fibration. This led us, with Dupont, to study the dynamics of endomorphisms of $\mathbb{P}^k$ which preserve a fibration. The main purpose of [DT18b] (see Chapter 3) is to generalize Jonsson's results [Jon99] on polynomial skew products to a broader framework. In particular, we obtain a structure result for the Green currents which can be helpful to study specific examples and bifurcation phenomena.

Presented publications

- *Speed of convergence towards attracting sets for endomorphisms of* $\mathbb{P}^k$. Indiana Univ. Math. Journal, 62:33–44, 2013.
- *Codimension one attracting sets in* $\mathbb{P}^k$. with Sandrine Daurat. Ergodic Theory and Dynam. Systems 38 :63–80, 2018.
- *Bifurcations in the elementary Desboves family.* with Fabrizio Bianchi. Proc. Amer. Math. Soc. 145 :4337–4343, 2017.
- *Attracting Currents and Equilibrium Measures for Quasi-attractors of* $\mathbb{P}^k$. Invent. math., 213:83–137, 2018.
- *Blenders near polynomial product maps of* $\mathbb{C}^2$. Journal of the European Math. Soc., to appear, 2017.
- *Dynamics of fibered endomorphisms of* $\mathbb{P}^k$. with Christophe Dupont, Ann. Scuola Normale Sup. Pisa, to appear, 2018.

Chapter 1

Attractors in $\mathbb{P}^k$

This chapter covers the works [Taf13], [DT18a], [Taf18] and some examples contained in [Taf17]. It deals with different kinds of "attractors" which are natural objects to consider in a dynamical setting. I start in Section 1.2 with their definitions and with simple examples in both smooth and complex dynamics. Section 1.3 is devoted to general results on attractors (or more precisely, on attracting sets and quasi-attractors) of $\mathbb{P}^k$ which can be obtained using pluripotential theory. In particular, it contains the main results of [Taf18] and a brief summary of the known results when this work began (see Section 1.3.3). Section 1.4 and Section 1.5 consider the frameworks of [DT18a] and [Taf13] respectively. Finally, Section 1.6 gives most of the examples I know.

1.1 Prologue

When considering a dynamical system $f\colon X \to X$, two natural questions arise:

(1) What is the asymptotic behavior of the orbit $(f^n(x))_{n\geq 0}$ for most of the points $x \in X$?

(2) What are the effects on the system of a small perturbation of f?

One aspect of the latter is to understand when f is *structurally stable,* i.e. under which conditions on f, for each sufficiently small (in a given topology) perturbation f_ϵ of f there exists a homeomorphism h_ϵ of X such that $f = h_\epsilon^{-1} \circ f_\epsilon \circ h_\epsilon$?

For endomorphisms of the projective space $\mathbb{P}^k$, these two questions are fairly well understood when $k = 1$. To a rational mapping $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ is associated a partition

$$\mathbb{P}^1 = F_f \cup J_f$$

where F_f is the *Fatou set* of f and J_f its *Julia set.* On the open set F_f the dynamics is not chaotic and it can be completely classified (see Section 1.2.3). On the compact set J_f the dynamics is chaotic but there exist several characterizations of this set giving many information on the geometry and the dynamics of J_f.

In the same way, if $(f_\lambda)_{\lambda\in M}$ is a holomorphic family of rational maps then Mañé-Sad-Sullivan [MSS83] and independently Lyubich [Lyu83b] introduced a partition

$$M = \mathrm{Stab} \cup \mathrm{Bif}.$$

The set Stab can be defined as the largest open subset of M where f_λ restricted to its Julia set is structurally stable with respect to perturbations in the family (see Section 2.1

for more details). This is called *J-stability*. The complementary Bif of Stab is called the *bifurcation locus* of the family $(f_\lambda)_{\lambda\in M}$. The key result about this partition is that Stab is always dense in M. Moreover, McMullen and Sullivan [MS98] proved that on an open and dense subset of Stab the structural stability on J_{f_λ} extends to the whole $\mathbb{P}^1$, i.e. in every family $(f_\lambda)_{\lambda\in M}$ structural stability is dense.

In higher dimension, the dichotomy between the Fatou set and the Julia set still exists but several new phenomena appear and a description as detailed as in dimension one is no longer possible. My approach is based on the following standard objects in smooth dynamics.

1.2 Attracting sets and chain recurrence classes

Since the beginning of the study of dynamical systems, the different possible notions of attractors have played a central role (see e.g. [Mil85]). Unlike my work during my thesis (see [Taf10]), I am now interested in the following topological objects.

Let $f\colon X\to X$ be a continuous self-map of a compact metric space. A non-empty subset A of X is an *attracting set* if it admits an open neighborhood U (called a *trapping region*) such that

$$\overline{f(U)}\subset U \quad \text{and} \quad A=\bigcap_{n\geq 0} f^n(U).$$

In particular, A is a compact invariant set for f, $f(A)=A$, and all orbits starting in U accumulate on some subset of A. So, such a set can be seen as a piece of the system and it is natural to study those which are *minimal* in some sense: they can be considered as the most elementary pieces. One way to ensure this is to consider attracting set A such that $f_{|A}$ is topologically transitive, i.e. $f_{|A}$ has a dense orbit. In this case, A is called an *attractor*. Unfortunately, this notion is very strong and many dynamical systems have no such attractor. Another way is to define, following [Hur82], *quasi-attractors* as decreasing intersections of attracting sets. In that setting, the *minimal quasi-attractors* are simply the ones which are minimal for the inclusion. Thanks to Zorn's lemma, any dynamical system defined on a compact space admits at least one minimal quasi-attractor. Notice that this terminology is not uniform in the literature. For example, minimal quasi-attractors here correspond to *attractors* in Ruelle's book [Rue89].

These objects are closely related to the *chain recurrent set* defined by Conley [Con78]. For $\epsilon>0$, a sequence $(x_i)_{0\leq i\leq n}$ is called an *ϵ-pseudo-orbit* between x and y in X if $n\geq 1$, $x_0=x$, $x_n=y$ and for all $0\leq i<n$, $\mathrm{dist}(f(x_i),x_{i+1})<\epsilon$. We say that $x\succ y$ if for all $\epsilon>0$ there exists an ϵ-pseudo-orbit between x and y. Then, the chain recurrent set is defined by

$$\mathscr{R}(f):=\{x\in X\,|\,x\succ x\}.$$

This is a closed invariant set which contains the non-wandering set and thus all the periodic orbits. Moreover, $\succ$ is a preorder on $\mathscr{R}(f)$ and the equivalence classes associated to it are called the *chain recurrence classes*: if $x\in\mathscr{R}(f)$ then its chain recurrence class $[x]$ consists of all $y\in\mathscr{R}(f)$ such that $x\succ y$ and $y\succ x$. The relation $\succ$ becomes an order on the classes by saying that $[x]\succ[y]$ if $x\succ y$.

Remark 1.2.1. *Another classical closed invariant set is the* non-wandering *set* $\Omega(f)$ *defined as the set of points* $x\in X$ *such that for every neighborhood* V *of* x *there exists* $n\geq 1$ *such that* $f^n(V)\cap V\neq\varnothing$. *The structural stability restricted to the non-wandering set is called the* Ω-stability. *When* X *is a smooth manifold, it was proven by Smale [Sma65] and Palis [Pal88] that* f *is* Ω*-stable with respect to* $\mathcal{C}^1$*-perturbations if and only if* f *is* Axiom

A (i.e. $\Omega(f)$ is a hyperbolic set for f and the periodic points are dense in $\Omega(f)$) and there is no cycle in the decomposition of $\Omega(f)$ in basic pieces. As observed in [BC04], a simpler way to state these theorems is

$$f \text{ is } \Omega\text{-stable in the } \mathcal{C}^1\text{-topology} \iff \mathscr{R}(f) \text{ is a hyperbolic set for } f$$

Notice that on $\mathbb{P}^1$, the classification of Fatou components implies easily that Ω-stability is equivalent to J-stability. One major open problem in one variable complex dynamics is wether this stability implies the hyperbolicity of the dynamics on the Julia set.

The chain recurrence classes are also closed invariant sets and they are related to the attracting sets and quasi-attractors in the following way. If $x \in X$ and $\epsilon > 0$ then the open set

$$U_{x,\epsilon} := \{y \in X \mid \text{there exists an } \epsilon\text{-pseudo-orbit between } x \text{ and } y\} \tag{1.1}$$

is a trapping region such that the attracting set $A_{x,\epsilon} := \cap_{n\geq 0} f^n(U_{x,\epsilon})$ contains all the chain recurrence classes $[y]$ with $x \succ y$ (notice that constructions similar to the one of $U_{x,\epsilon}$ will play a decisive role in Section 1.3.8). In particular, if $x \in \mathscr{R}(f)$ then $[x]$ is contained into the quasi-attractor $A_x := \cap_{\epsilon>0} A_{x,\epsilon}$. Moreover, if the *basin* of an attracting set A is defined by $\mathcal{B}_A := f^{-n}(U)$ (where U is a trapping region for A) then

$$[x] = A_x \setminus \left(\bigcup \mathcal{B}_A\right), \tag{1.2}$$

where the union is taken over all attracting sets A such that $x \notin A$. A direct consequence of these observations is the following important fact:

> *The minimal quasi-attractors correspond exactly to the chain recurrence classes minimal for $\succ$.*

Therefore, as my main results in the subject are about quasi-attractors, they also hold for minimal chain recurrence classes. But, some of these results are not true for arbitrary classes and very little is known about them in complex dynamics.

Let me point out that the observations above could be summarized through a *Lyapunov function*, i.e. a continuous function $\phi \colon X \to \mathbb{R}$ which is

- strictly decreasing on the orbits of $X \setminus \mathscr{R}(f)$,
- satisfies that for all $x, y \in \mathscr{R}(f)$, $\phi(x) = \phi(y)$ is equivalent to $[x] = [y]$,
- and $\phi(\mathscr{R}(f))$ is a totally discontinuous compact subset of $\mathbb{R}$.

The sublevel sets of such a function give a family of trapping regions which separate the different chain recurrence classes. Conley proved in [Con78] that Lyapunov functions always exist for continuous maps on a compact metric space.

To conclude this part, notice that the objects introduced above are directly related to the questions starting this section:

- the attracting sets catch a full open set of orbits,
- the ϵ-pseudo-orbits can be seen as the orbits of a randomly perturbed system.

1.2.1 Adding machine in smooth dynamics

The most basic examples of attracting sets are the whole space X (as X is compact) and *attracting cycles* (i.e. attractive periodic points, also often called *sinks*). An arbitrary attracting set can be much more complicated but there exist some basic limitations. In particular, the inclusion $\overline{f(U)} \subset U$ implies that f admits at most countably many attracting sets and each of them has finitely many connected components. The example below shows that this is no longer the case for quasi-attractors. As a consequence of Theorem 1.3.1, such an example cannot exist for endomorphisms of $\mathbb{P}^k$.

Let $D \subset \mathbb{R}^2$ be the open unit disk and let D_1, D_2 be two topological disks such that $\overline{D_1 \cup D_2} \subset D$ and $\overline{D_1} \cap \overline{D_2} = \varnothing$. Let f be a continuous self-map of D such that

$$\overline{f(D)} \subset D, \quad \overline{f(D_1)} \subset D_2 \quad \text{and} \quad \overline{f(D_2)} \subset D_1.$$

Thus, D and $D_1 \cup D_2$ are two trapping regions for f. The attracting set associated to D is connected and the one associated to $D_1 \cup D_2$ has two connected components. Moreover, it is possible to choose f such that, for $i \in \{1, 2\}$, $f^2_{|D_i}$ is conjugated to $f_{|D}$. In this situation, f has a decreasing sequence of attracting sets $(A_n)_{n\geq 0}$ such that A_n has 2^n connected components. The quasi-attractor $K := \cap_{n\geq 0} A_n$ has infinitely many connected components. The map f can be chosen such that these components are points and in this case, K is a minimal quasi-attractor such that $f_{|K}$ is an *adding machine*. This implies that

1) K is a Cantor set,

2) $f_{|K}$ is minimal and uniquely ergodic, in particular it has no periodic orbits,

3) $f_{|K}$ has zero topological entropy,

4) for each $n \geq 1$, $f^n_{|K}$ has 2^n minimal quasi-attractor and there is no mixing measure for $f^n_{|K}$.

The construction above gives a continuous map. It is possible, even though much more delicate, to fulfill it for $\mathcal{C}^\infty$ maps. Finally, observe that if there exists a second pair of disks $B_1, B_2 \subset D$ such that $\overline{f(B_1)} \subset B_2$, $\overline{f(B_2)} \subset B_1$, and $f^2_{|B_i}$ is conjugated to $f_{|D}$ then f admits uncountably many minimal quasi-attractors as above, one for each infinite sequence in $\{B, D\}$. See [BD02a] for the abundance of this phenomenon in the $\mathcal{C}^1$ topology.

1.2.2 Attracting sets in $\mathbb{C}^k$

As I said in the Introduction, the only possible attracting sets in $\mathbb{C}^k$ are finite unions of attracting cycles. This can be deduced from [Tsu81] but the proof is elementary. Notice that one can hope to use similar arguments in the setting of Section 1.3 (attracting sets in $\mathbb{P}^k$) providing a good counterpart to Montel's theorem in the space $\mathscr{C}_p(U)$ of positive closed (p, p)-currents in U. However, this last point seems delicate (see Remark 1.3.32).

Proposition 1.2.2. *Let $U \subset \mathbb{C}^k$ be an open set and let $f : U \to U$ be a holomorphic map such that $\overline{f(U)}$ is a compact subset of U. Then, the attracting set $A := \cap_{n\geq 0} f^n(U)$ is a finite union of attracting cycles.*

Proof. First, observe that since A is compact, it is contained in a finite union of connected components of U and if this union is taken to be minimal then f induces of permutation on this set of components. Hence, possibly by replacing f by a large iterate, we can assume that U is connected and also bounded since $f(U)$ is bounded.

Therefore, Montel's Theorem on uniformly bounded holomorphic functions implies that the family $\{f^n\}_{n\geq 1}$ si *normal* i.e. every sequence in this family has a subsequence converging, locally uniformly, to a holomorphic map from U to $\mathbb{C}^k$. Let $(f^{n_i})_{i\geq 1}$ be such a sequence converging to $g\colon U \to \mathbb{C}^k$ where $(n_i)_{i\geq 1}$ is increasing. In particular, $l_i := n_{i+1}-n_i \geq 1$ and by normality, up to a subsequence, we can assume that $(f^{l_i})_{i\geq 1}$ converge to a map $h\colon U \to \mathbb{C}^k$. Actually, $n_i \geq 1$ and $l_i \geq 1$ imply that $g(U)$ and $h(U)$ are contained in $\overline{f(U)}$ which is a compact subset of U. Hence, $h \circ g$ is well defined and satisfies $h \circ g = g$ on U, i.e. $g(U)$ is contained in the set of fixed points of h. But this set is a closed analytic subset of U, contained in the compact set $\overline{f(U)}$. Thus, it is a compact analytic subset of $\mathbb{C}^k$ and so it is a finite set. As $g(U)$ is connected, $g(U)$ is reduced to a point $x_0 \in U$.

Since g commutes with f, $f(x_0) = f(g(x_0)) = g(f(x_0)) = x_0$ and x_0 is a fixed point of f. Therefore, every subsequence of $(f^n)_{n\geq 1}$, and thus the full sequence, must converge to the constant function equal to x_0. This implies that the differential $D_{x_0}f$ satisfies $\lim_{n\to\infty}(D_{x_0}f)^n = 0$, i.e. x_0 is an attractive fixed point and U is contained in its basin. □

Notice that the boundedness of $f(U)$ is essential: a complex Hénon maps in $\mathbb{C}^2$ possesses an open set $U \subset \mathbb{C}^2$ satisfying $\overline{f(U)} \subset U$ while $\cap_{n\geq 0} f^n(U)$ is not trivial.

1.2.3 Situation on $\mathbb{P}^1$

The classification of attracting sets and chain recurrence classes of a rational mapping f of $\mathbb{P}^1$ follows easily from Proposition 1.2.2 and Montel Theorem. It can also be deduced from the classical Fatou/Julia theory.

For attracting sets, there are two possibilities. If A is an attracting set of $\mathbb{P}^1$ with a trapping region U then either

- $A = U = \mathbb{P}^1$,
- or $U \neq \mathbb{P}^1$ and $\overline{f(U)}$ can be seen as a compact set of $\mathbb{C}$ and by Proposition 1.2.2, A is a finite union of attracting cycles.

In particular, all the quasi-attractors are attracting sets. Hence, using (1.2), a chain recurrence class is either

- an attracting cycle,
- or equal to $\mathbb{P}^1 \setminus (\cup \mathcal{B}_A)$ where the union is taken over all attracting cycles.

This last chain recurrence class is actually related to the Fatou/Julia decomposition. Recall that the Fatou set F_f of f is the largest open set where the family of iterates $\{f^n\}_{n\geq 1}$ is *normal* (or, equivalently, equicontinuous). Its complementary $J_f := \mathbb{P}^1 \setminus F_f$ is a non-empty compact set called the Julia set and, by definition, if V is a neighborhood of $x \in J_f$ then the family $\{f^n_{|V}\}_{n\geq 1}$ is not normal. This implies, by Montel's Theorem on holomorphic maps omitting three values in $\mathbb{P}^1$, that $\mathbb{P}^1 \setminus (\cup_{n\geq 1} f^n(V))$ contains at most two points. Hence, we have the following basic result.

Proposition 1.2.3. *If $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ is a rational mapping and $x \in J_f$ then for all $y \in \mathbb{P}^1$, $x \succ y$. In particular, J_f is contained in a unique chain recurrence class $[J_f] := \mathbb{P}^1 \setminus (\cup \mathcal{B}_A)$ and this class is maximal with respect to $\succ$.*

In particular, if f has no attracting cycle then $[J_f] = \mathbb{P}^1$ and f is *chain transitive*, i.e. f has a unique chain recurrence class which is equal to the whole space.

Another way to obtain the same result is to use the classification of the dynamics on F_f. A remarkable theorem of Sullivan [Sul85] states that if Ω is a connected component of F_f (also called a *Fatou component*) then there exist $p > q \geq 0$ such that $f^p(\Omega) = f^q(\Omega)$, i.e. every Fatou component of a rational mapping of $\mathbb{P}^1$ is pre-periodic. Moreover, the invariant Fatou components are classified. There are four possibilities:

- the basin of an *attracting fixed point*,
- the basin of a *parabolic fixed point*,
- a *Siegel disk*, i.e. a topological disk on which f is conjugated to an irrational rotation,
- a *Herman ring*, i.e. a topological annulus on which f is conjugated to an irrational rotation.

On the other hand, the dynamics on J_f is transitive and all parabolic periodic points belong to J_f. Hence, J_f is contained in a unique class $[J_f]$ and the other classes are attracting cycles. The class $[J_f]$ is equal to the union of J_f and of all the Fatou components eventually map to a parabolic basin, a Siegel disk or a Herman ring. In particular, $[J_f] \neq J_f$ when f has such a component, for example for $f(z) = z^2 + 1/4$.

Example 1.2.4. *If $f(z) = z^2 + c$ where $c \in \mathbb{C}$ has sufficiently large modulus then F_f has only one component which is the basin of ∞ and J_f is a Cantor set. Hence, the chain recurrent set is $\mathscr{R}(f) = J_f \cup \{\infty\}$, each piece corresponding to a chain recurrence class. Observe that the one associated to J_f has infinitely many connected components.*

1.3 Quasi-attractors in $\mathbb{P}^k$ and attracting currents

The main aim of this section is to explain the results of [Taf18]. Many of these results are heavily based on pluripotential theory so we start with a "current-free" summary.

1.3.1 Main results without currents

Theorem 1.3.1. *Let f be a holomorphic endomorphism of $\mathbb{P}^k$ of degree $d \geq 2$. If K is a minimal quasi-attractor of f then there exist two integers $0 \leq s \leq k$ and $n_0 \geq 1$ such that*

- *the topological entropy of $f_{|K}$ is $s \log d$,*
- *K is the disjoint union of connected minimal quasi-attractors $K_1, \dots, K_{n_0}$ for f^{n_0},*
- *on each K_i, there exists a measures ν_i which is mixing for f^{n_0}, of maximal entropy $s \log(d^{n_0})$ on K_i and with at least s positive Lyapunov exponents.*

Remark 1.3.2. *In general, f^{n_0} is not topologically transitive on K_i. Indeed, if f is a rational mapping of $\mathbb{P}^1$ such that all Fatou components are eventually mapped to Siegel disks then $K = \mathbb{P}^1$ is a minimal quasi-attractor such that $s = n_0 = 1$ and f is not topologically transitive (when the Fatou set is not empty).*

Remark 1.3.3. *Since minimal chain recurrence classes correspond to minimal quasi-attractors, Theorem 1.3.1 applies to these classes. In particular, such a class has finitely many connected components. As Example 1.2.4 shows, this doesn't hold for an arbitrary class. I don't know if it holds for arbitrary quasi-attractors.*

Observe that this theorem is the counterpart for minimal quasi-attractors of Theorem 0.0.1 in the introduction. Indeed, the latter is a direct consequence of the former as an attractor is a minimal quasi-attractor. However, the finiteness of the number of connected components n_0 is obvious for attractors (and true for continuous maps) while it is an important point of Theorem 1.3.1.

Another important point is the meaning of the integer s. This invariant was introduced by Daurat [Dau14] as the *dimension* of K. Her definition relies on pluripotenrial theory (see Section 1.3.5) and is independent of f. In particular, the topological entropy of $f_{|K}$ only depends on the geometry of K and on the degree of f. This dimension can also be characterized in terme of Julia sets which gives the following consequences.

Corollary 1.3.4. *Let f, K and s be as in Theorem 1.3.1.*

- *If s is maximal (i.e. $s = k$) then $K = \mathbb{P}^k$.*
- *If $s = 0$ (i.e. $f_{|K}$ has zero topological entropy) then K is an attracting cycle.*

Here is a less direct consequence of the proof of Theorem 1.3.1 (which is based on the local finiteness of attracting currents).

Corollary 1.3.5. *Let f be a holomorphic endomorphism of $\mathbb{P}^k$. Then, the set of minimal quasi-attractors of f is at most countable.*

This set can be countable when $k \geq 2$ as shown the examples of Gavosto [Gav98] (see also [Buz97]) of endomorphisms of $\mathbb{P}^2$ with infinitely many sinks coming from Newhouse phenomenon. And the minimality assumption in this corollary is important since there exist endomorphisms with uncountably many quasi-attractors (see Theorem 1.6.8).

All the results above show that minimal quasi-attractors of $\mathbb{P}^k$ are subject to much more constraints that what can happen in smooth dynamics. In particular, the points **1)**, **3)**, **4)** of Section 1.2.1 on adding machines and uncountable sets of minimal quasi-attractors cannot exist for holomorphic endomorphisms of $\mathbb{P}^k$.

As we will see in Section 1.3.3, some of the results above, in particular Corollary 1.3.5, can be deduced when $k = 2$ from [FW99] using totally different techniques. And a key inspiration for Theorem 1.3.1 was the work of Dinh [Din07] where he obtained a similar result for attracting sets possessing a trapping region with special geometric properties. Before reviewing these works, we need to recall some classical facts about pluripotential theory and dynamics of holomorphic endomorphisms of $\mathbb{P}^k$.

1.3.2 Basic facts on pluripotential theory and dynamics on $\mathbb{P}^k$

The introduction of pluripotential theory (i.e. plurisubharmonic functions and positive currents) was a turning point in complex dynamics in several variables (see especially [BS91a] and [FS95a]). A posteriori, almost thirty years later, the introduction of the actions of f on currents seems natural. Indeed, it is usual to study the dynamics of $f\colon X \to X$ through its actions on other spaces (trees, partitions, vector spaces etc.). If X is a compact complex manifold and f is holomorphic then one can consider the space of positive currents, on which f acts by push-forward and pull-back. This space is a convexe cone whose structure is far from being well-understood but it has some good properties, with respect to compactness in particular. Moreover, these actions (restricted to closed currents) are related to push-foward and pull-back operators induced by f on the cohomology groups of X, which are especially simple when $X = \mathbb{P}^k$. More generally, there are deep interplays between the dynamics of f, the geometry of X and the complex analytic tools provide by pluripotential theory. Actually, in the proofs of the results of this section, the crucial assumption that f is holomorphic will mainly take part by way of these actions on currents. Here is a brief illustration of how these actions will be used in what follows.

One feature of ergodic theory is to study Birkhoff sums

$$\frac{1}{N}\sum_{n=1}^{N}(f^n)_*\delta_x, \tag{1.3}$$

(where $(f^n)_*\delta_x$ is the push-forward of the Dirac mass at x by f^n and is equal to $\delta_{f^n(x)}$) instead of orbits $(f^n(x))_{n\geq 1}$. The space of positive measures has good compactness properties and every limit value ν of (1.3) is invariant, $f_*\nu = \nu$. However, apart from this invariance, almost nothing can be said in general on ν. Positive currents allow to generalized this idea to orbits of complex analytic sets. To such a set V is associated a *current of integration on* V, denoted by $[V]$, and we can consider sums of the form

$$\frac{1}{N}\sum_{n=1}^{N}c_n(f^n)_*[V], \tag{1.4}$$

where $c_n > 0$ are well chosen renormalization constants and $(f^n)_*[V]$ is the push-forward of $[V]$ by f^n (which is often simply $[f^n(V)]$). When V is reduced to the point x, (1.4) with $c_n = 1$ corresponds exactly to (1.3). Moreover, the space of positive currents has the same compactness properties than the space of positive measures and every limit value of (1.4) is a positive current invariant by f_* (up to a constant). Under some circumstances (for example if V is a *closed* analytic set) these limit values are *closed* as currents. And positive closed currents (those which are not measures) are very rigid objects. As we will see later, the simple fact that a subset $A \subset \mathbb{P}^k$ supports such a current puts strong geometric contrains on A and on $\mathbb{P}^k \setminus A$ and gives tools to study the dynamics on A.

We now give more detailed definitions and results. We refer to [Dem12] for an exposition on pluripotential theory and to [Sib99] and [DS10a] for surveys on its interactions with complex dynamics.

Positive closed currents

Let X be a complex manifold of dimension k and let $0 \leq p, q \leq k$ be two integers. For the sequel, we need to defined three notions:

- currents of bidegree (p, q), or equivalently of bidimension $(k-p, k-q)$,
- closed currents,
- positive currents.

(p,q)-currents: A *smooth differential form* ϕ *of bidegree* (p, q) on X (or simply a (p, q)-form) is a form which can be written in local coordinates $(z_1, \ldots, z_k)$ as

$$\phi = \sum_{|I|=p,|J|=q}\phi_{I,J}dz_I \wedge d\overline{z}_J,$$

where $\phi_{I,J}$ are smooth functions, $I = (i_1, \ldots, i_p)$ (resp. $J = (j_1, \ldots, j_q)$) is a multi-index of length p (resp. q) and $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, $d\overline{z}_J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$. A *current of bidegree* (p, q) (or a (p, q)-current) is a continuous linear form on the space of smooth differential forms of bidegree $(k-p,k-q)$ with compact support in X. Such a current S can be written in local coordinates as

$$S = \sum_{|I|=p,|J|=q} S_{I,J} dz_I \wedge d\overline{z}_J, \tag{1.5}$$

where the coefficients $S_{I,J}$ are distributions. A form or a current is of *bidimension* (p,q) if it is of bidegree $(k-p,k-q)$. As we will see in Example 1.3.6, the bidimension of the current of integration associated to an analytic subset $V \subset X$ of dimension p is (p,p).

The topology that we use on currents is the weak topology: a sequence S_n converges to S if for every from ϕ with compact support, $\langle S_n, \phi\rangle$ converges to $\langle S, \phi\rangle$.

Closed currents: By definition, a current S of bidegree (p,q) is a pairing $\phi \mapsto \langle S, \phi\rangle$ with $(k-p,k-q)$-forms and we say that S is *closed* if $\langle S, \phi\rangle = 0$ as soon as ϕ is *exact* i.e. $\phi = d\psi$ for some differential form ψ.

Positive currents: Positivity is only defined for (p,q)-current with symmetric bidegree, i.e $p = q$. Roughly speaking, a current S of bidegree (p,p) is positive if for every complex submanifold V of dimension p, the restriction of S to V is a positive measure. However, the restriction of a current to a submanifold is not necessarily well-defined and even when it is the case (for smooth forms for example), the resulting notion is not stable under wedge products. The above idea corresponds to weak positivity and we need a slightly stronger notion.

A (p,p)-current S on X is *weakly positive* if for every family $(\alpha_j)_{1\leq j\leq k-p}$ of $(1,0)$-forms with compact support,

$$\langle S, (i\alpha_1 \wedge \overline{\alpha_1}) \wedge \cdots \wedge (i\alpha_{k-p} \wedge \overline{\alpha_{k-p}})\rangle \geq 0.$$

A (p,p)-current S is *positive* if for every weakly positive $(k-p,k-p)$-form ϕ with compact support in X, $\langle S, \phi\rangle \geq 0$.

Notice that in maximal bidegree, a (k,k)-current is always closed and can be identified to a distribution. In particular, positive (k,k)-currents correspond to positive measures.

Beyond technical definitions, the following example is essential.

Example 1.3.6. *Let Y be a closed complex submanifold of X of dimension p and let V be an open subset of Y. The* current of integration $[V]$ *associated to V is a current of bidegree $(k-p,k-p)$ (and of bidimension (p,p)) defined by*

$$\langle [V], \phi\rangle := \int_V \phi,$$

for every (p,p)-form ϕ with compact support. This current is positive and if $V = Y$ (i.e. V is a closed submanifold) then $[V]$ is closed as current. This result has been generalized by Lelong [Lel57] to singular analytic sets: if Y is a closed analytic subset of X of pure dimension p then the current of integration on the regular part of Y defined a positive closed current on X, also denoted by $[Y]$.

All the currents which appear in what follows can be obtained as limits of convex combinations of such currents. In some sense, the currents that we consider encoded, locally on each open set, the asymptotic behavior of the volume of a sequence of analytic subsets.

Plurisubharmonic functions

Plurisubharmonic functions form an important class of functions in several variables complex analysis. On Riemann surfaces, this class coincides with the one of subharmonic functions. One way to define a plurisubharmonic function u in higher dimension is that u restricted to every one-dimensional complex manifold is a subharmonic function. Another way is to use the $\partial\overline{\partial}$ operator. For a $\mathcal{C}^2$ function $u\colon X \to \mathbb{R}$, plurisubharmonicity simply means that, in local coordinates $(z_1, \dots, z_k)$, the form

$$i\partial\overline{\partial}u := i \sum_{1\leq a,b\leq k} \frac{\partial^2 u}{\partial z_a \partial \overline{z_b}} dz_a \wedge d\overline{z_b},$$

is a positive $(1,1)$-form, i.e. at every point z the Hermitian matrix $(\partial^2 u/\partial z_a \partial\overline{z_b}(z))$ is semipositive. The operator $\partial\overline{\partial}$ can be extended to L^1_{loc}-functions and more generally to currents. The normalization $dd^c := \frac{i}{\pi}\partial\overline{\partial}$ is often used. And $u\colon X \to \mathbb{R}\cup\{-\infty\}$ is *plurisubharmonic* if

- u is upper semicontinuous and $u \in L^1_{\text{loc}}(X)$,
- $dd^c u$ is a positive $(1,1)$-current.

Moreover, the converse holds locally: if S is a positive closed $(1,1)$-current then on small open sets S is equal to $dd^c u$ where u is plurisubharmonic. One says that u is a *local potential* of S. The fact that positive closed $(1,1)$-currents have such potentials greatly simplifies their study. However, in what follows, we will mainly use plurisubharmonic functions through *structural varieties* of currents in any bidegree. Roughly speaking, a structural variety parametrized by a complex manifold M is a familiy of currents $(S_\theta)_{\theta\in M}$ such that

$$\theta \mapsto \langle S_\theta, \phi\rangle$$

is plurisubharmonic on M as soon as $dd^c\phi$ is a positive form (see Section 1.3.6 for more details).

Plurisubharmonic functions have many good properties. In particular, they verify the maximum principle and the mean value inequality. Rather than making an exhaustive list, here is a simple result which plays a central role in the study of quasi-attractors.

Lemma 1.3.7. *Let $(u_n)_{n\geq 0}$ be a uniformly bounded sequence of subharmonic functions defined on $\mathbb{D}$ which is locally equicontinuous on $\mathbb{D}^*$. Assume that there exists $c \in \mathbb{R}$ such that*

$$\limsup_{n\to\infty} u_n(\theta) \leq c,$$

for all $\theta \in \mathbb{D}$ and

$$\lim_{n\to\infty} u_n(0) = c.$$

Then the sequence $(u_n)_{n\geq 0}$ converges pointwise to the constant function c.

The above result will be combined to the following elementary one.

Lemma 1.3.8. *Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of real numbers. If there are two constants $c \in \mathbb{R}$ and $\alpha > 0$ such that*

$$\limsup_{n\to\infty} b_n \leq \alpha c, \quad \limsup_{n\to\infty}(a_n - b_n) \leq (1-\alpha)c \quad \textit{and} \quad \lim_{n\to\infty} a_n = c$$

then $\lim_{n\to\infty} b_n = \alpha c$.

Currents on $\mathbb{P}^k$ and dynamics

From now on, f is an endomorphism of $\mathbb{P}^k$ of degree $d \geq 2$ and $0 \leq p \leq k$ is an integer.

Let ω be the standard Fubini-Study form on $\mathbb{P}^k$ normalized such that the volume form ω^k has mass 1. This form gives a way to define *the mass* of a positive (p,p)-current S on $\mathbb{P}^k$ by $\|S\| := \langle S, \omega^{k-p} \rangle$ and we denote by $\mathscr{C}_p(\mathbb{P}^k)$ the set of all positive closed (p,p)-currents of mass 1 on $\mathbb{P}^k$. As for positive measures, thanks to the normalization of the mass, $\mathscr{C}_p(\mathbb{P}^k)$ is a compact set. And, for a subset E of $\mathbb{P}^k$ we define $\mathscr{C}_p(E)$ as the set of currents in $\mathscr{C}_p(\mathbb{P}^k)$ which are supported in E.

In what follows, we will extensively study the action by push-forward of f on $\mathscr{C}_p(E)$, for some specific E. The push-forward operator f_* on currents is simply defined by

$$\langle f_* S, \phi \rangle := \langle S, f^* \phi \rangle,$$

where f^* is the standard pull-back on differential forms. The pull-back of a current is more delicate to define but this can be done in our setting (see [DS07]). An important point about these actions is that the masses of $f_* S$ and $f^* S$ can be computed using cohomology. Actually, since the cohomology groups $H^{p,p}(\mathbb{P}^k, \mathbb{R})$ are unidimensional, the class of a positive closed current only depends on its mass. Moreover, by Bézout's Theorem the action of f^* on $H^{p,p}(\mathbb{P}^k, \mathbb{R})$ is multiplication by d^p and, by duality, the action of f_* on $H^{p,p}(\mathbb{P}^k, \mathbb{R})$ is multiplication by d^{k-p}. Hence, $d^{-p} f^*$ and $d^{-(k-p)} f_*$ define two operators from $\mathscr{C}_p(\mathbb{P}^k)$ to itself.

The dynamics induced by $d^{-p} f^*$ on $\mathscr{C}_p(\mathbb{P}^k)$ is quite special. Dinh and Sibony [DS09] showed that there exists a current T^p, called the *Green (p,p)-current of f*, such that if $S \in \mathscr{C}_p(\mathbb{P}^k)$ is smooth then

$$\lim_{n \to \infty} \frac{1}{d^{pn}} (f^n)^* S = T^p. \tag{1.6}$$

Moreover, the Green $(1,1)$-current T has continuous local potentials so its self-intersection of order p is well-defined and coincides with T^p. The support $\mathscr{J}_p$ of T^p is called the *Julia set of order p* of f and these sets define a filtration

$$\varnothing =: \mathscr{J}_{k+1} \subset \mathscr{J}_k \subset \cdots \subset \mathscr{J}_1 \subset \mathscr{J}_0 := \mathbb{P}^k.$$

By a result of Fornæss-Sibony [FS95a] and Ueda [Ued94], $\mathscr{J}_1$ is equal to the Julia set J_f of f, i.e. the complement of the largest open set where the family of iterates $(f^n)_{n \geq 1}$ is normal.

The set $\mathscr{J}_k$, sometimes called *the small Julia set*, is the support of the *equilibrium measure* of f, $\mu := T^k$, which is the unique measure of maximal entropy $k \log d$ of f and which has many other interesting dynamical properties (see, e.g., [FS94b, BD99, BD01, DS10b]). However, the measure μ gives no information on proper quasi-attractors since, by Proposition 1.3.11 below, a quasi-attractor intersecting $\mathscr{J}_k$ is automatically equal to $\mathbb{P}^k$.

1.3.3 Previously known results

Before [Taf18], many other works have been devoted to attractors or quasi attractors of holomorphic endomorphisms of $\mathbb{P}^k$. However, most of them were aimed at constructing interesting examples (e.g., [JW00, FS01, Ron12, Dau14, DT18a]) or were working with a set of assumptions, referred to as (HD) in what follows, first introduced by Dinh [Din07] (see [Taf13, Dau14, DT18a]). One exception is [FW99] where Fornæss and Weickert proved, in a rather simple way, the following result.

Theorem 1.3.9 ([FW99]). *Let f be a holomorphic endomorphism of $\mathbb{P}^k$. If K is a quasi-attractor of f which is not a finite union of attracting cycles then K contains* an entire curve*, i.e. the image of a non-constant holomorphic map $\phi\colon \mathbb{C} \to \mathbb{P}^k$.*

As easy consequences, the Hausdorff dimension of such quasi-attractor K is greater than or equal to 2 and it must intersect the Julia set of f. As we have seen in Theorem 1.3.1, K supports an ergodic measure with at least one positive Lyapunov exponent which gives an alternative way to obtain entire curves contained in K. Observe that K can be large enough to contain many entire curves which are not related to the dynamics (see Theorem 1.6.9 for such an example with $K \neq \mathbb{P}^k$).

When $k = 2$, using a cohomological argument or the pseudoconvexity of $\mathbb{P}^2 \setminus \overline{\phi(\mathbb{C})}$, Theorem 1.3.9 gives

Corollary 1.3.10 ([FW99]). *If f is an endomorphism of $\mathbb{P}^2$ then it admits at most one minimal quasi-attractor K which is not an attracting cycle. In particular, K is a minimal quasi-attractor for every iterate of f and it is connected. Moreover, $\mathbb{P}^2 \setminus K$ is pseudoconvex.*

Notice that Example 1.6.2 shows that this uniqueness result is no longer true in higher dimension.

Another way to obtained results on quasi-attractors is to use convergences toward the Green currents similar to (1.6). For example, Fornæss-Sibony [FS94b] proved that almost every $y \in \mathbb{P}^k$ satisfy

$$\lim_{n\to\infty} \frac{1}{d^{kn}} \sum_{f^n(a)=y} \delta_a = \mu.$$

where μ is the equilibrium measure of f. This counterpart in higher dimension to Proposition 1.2.3 follows easily.

Proposition 1.3.11. *Let x be in the small Julia set $\mathscr{J}_k$ of f. Then for all $y \in \mathbb{P}^k$, $x \succ y$. In particular, $\mathscr{J}_k$ is contained in a unique chain recurrence class $[\mathscr{J}_k]$ and this class is maximal with respect to $\succ$. Moreover, a quasi-attractor intersecting $\mathscr{J}_k$ has to be equal to $\mathbb{P}^k$.*

Remark 1.3.12. *A non-trivial automorphism of $\mathbb{C}^2$ of degree $d \geq 2$ also has an equilibrium measure μ of entropy $\log d$ which is mixing so it corresponds to a unique chain recurrence class $[J^*]$ where $J^* := \operatorname{supp}(\mu)$. Bedford and Smilie have proved [BS91b] that the chain recurrent set of such automorphism is the union of $[J^*]$ with attracting/repelling cycles.*

Another consequence of equidistribution is the following result obtained by Sibony [Sib99]. It gives a first hint toward the different equivalent formulations of the dimension of a quasi-attractor given by Daurat (see Section 1.3.4).

Theorem 1.3.13 ([Sib99]). *Let $K \subset \mathbb{P}^k$ be a quasi-attractor for f and let $0 \leq s \leq k$. If the Hausdorff dimension of K is strictly smaller than $2s$ then $K \cap \mathscr{J}_s = \varnothing$.*

In particular, if this Hausdorff dimension is strictly smaller than 2 then K is a union of attracting cycles, which was also a consequence of Theorem 1.3.9.

1.3.4 Main results on attracting currents

The previous section shows that if $K \subset \mathbb{P}^k$ is a quasi-attractor which is not an attracting cycle then there are several constraints on the geometry of K and $\mathbb{P}^k \setminus K$. Using the theory of currents, this section will go further in this direction.

In [Din07], Dinh constructed, under geometric assumptions which will be referred to as (HD) is what follows, an *attracting current* and an *equilibrium measure* associated to an attracting set. My main motivation in [Taf18] was to generalize this construction to all quasi-attractors without any assumption. A first step to this end is to obtained some geometrical information on quasi-attractors and trapping regions in $\mathbb{P}^k$. This can be done using the notion of *dimension* introduced by Daurat [Dau14] for attracting sets.

Definition 1.3.14 ([Dau14])**.** *Let $A \subset \mathbb{P}^k$ be an attracting set and let $0 \leq s \leq k$ be an integer. The* dimension *of A is s if $\mathscr{C}_{k-s}(A) \neq \varnothing$ and $\mathscr{C}_{k-s-1}(A) = \varnothing$. The dimension of a trapping region is by definition the dimension of the associated attracting set.*

In particular, a result of Federer [Fed69] on the support of flat currents gives the following lower bound to the Hausdorff dimension $\dim_{\mathscr{H}}(A)$ of A: if A is an attracting set of dimension s then $\dim_{\mathscr{H}}(A)$ is larger than or equal to $2s$. However, there is no non-trivial upper bound to $\dim_{\mathscr{H}}(A)$. Actually, Theorem 1.6.9 gives an example of an attractor $A \subset \mathbb{P}^2$ of dimension 1 with non-empty interior and thus with Hausdorff dimension equal to 4.

Since a quasi-attractor K is a decreasing intersection of attracting sets $(A_i)_{i\geq 1}$, one can define the dimension of K to be the dimension of A_i, for i large enough. A key point about this definition is the following equivalent characterization.

Proposition 1.3.15. *A quasi-attractor K has dimension s if and only if*

$$K \cap \mathscr{J}_s \neq \varnothing \quad \text{and} \quad K \cap \mathscr{J}_{s+1} = \varnothing.$$

Although simple, this result is essential since it implies that K always satisfies one of the consequence of the assumptions (HD) of Dinh. In Technical words, since $K \cap \mathscr{J}_{s+1} = \varnothing$, the Green current T^{s+1} belongs to $\mathscr{C}_{s+1}(\mathbb{P}^k \setminus K)$ which implies that K is weakly $(k-s)$-pseudoconvex (see Section 1.3.6). This will allow us to use the dd^c-method developed by Dinh and Sibony to study the dynamics induced by f on $\mathscr{C}_{k-s}(U)$ for a trapping region U of dimension s containing K. In particular, we obtain that there exists on each attracting set at least one current which exhibits equidistribution properties.

Theorem 1.3.16. *Let A be an attracting set of dimension s with a trapping region U. There exist a trapping region $D_\tau \subset U$ and a current τ in $\mathscr{C}_{k-s}(D_\tau)$ such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{d^{ns}} (f^n)_* R = \tau$$

for all continuous currents R in $\mathscr{C}_{k-s}(D_\tau)$.

As this type of currents will have a central role in what follows, we coin the following definition.

Definition 1.3.17. *Let V be a trapping region of dimension s. A current $S \in \mathscr{C}_{k-s}(V)$ is* attracting *on V if*

$$\lim_{n\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{d^{ns}} (f^n)_* R = S,$$

for all continuous form R in $\mathscr{C}_{k-s}(V)$. A current in $\mathscr{C}_{k-s}(\mathbb{P}^k)$ is an attracting current *if it is attracting on some trapping region of dimension s.*

These currents can be considered as attractive periodic points in the set $\mathscr{C}_{k-s}(\mathbb{P}^k)$. By definition, if $S \in \mathscr{C}_{k-s}(V)$ is attracting on V then it is the unique attracting current in $\mathscr{C}_{k-s}(V)$. However, there can exist several such currents supported on an attracting set A and it is not possible to remove the Cesàro mean in the theorem. Here is a trivial example of this. Let $A = \{p_0, p_1, p_2\}$ be the union of an attracting fixed point p_0 and an attracting cycle $\{p_1, p_2\}$ of period 2. Then A is an attracting set of dimension 0 with two attracting measures, the Dirac mass δ_{p_0} and $2^{-1}(\delta_{p_1} + \delta_{p_2})$. For the latter, if R is a smooth probability supported in a small neighborhood of p_1 then the sequence $(f^n)_*R$ has two different limit values, δ_{p_1} and δ_{p_2}. However, if we replace f by f^2 in this example then the sequence $(f^n)_*R$ converges and we obtain three attracting measures which cannot be decomposed anymore. On feature of Theorem 1.3.18 below is that this phenomenon still holds in the general case.

The non-uniqueness of the attracting current in Theorem 1.3.16 is an important issue in order to study a quasi-attractor $K = \cap_{i\geq 1} A_i$. Each attracting set A_i supports an attracting current but them might be all different. Theorem 1.3.18 also solves this problem.

Theorem 1.3.18. *Let A be an attracting set of dimension s. The set of attracting currents of bidimension (s, s) supported in A is finite. Moreover, there exists an integer $n_0 \geq 1$ such that if τ is an attracting current of bidimension (s, s) supported in A for f^{n_0} then*

$$\lim_{n\to\infty} \frac{1}{d^{nn_0 s}} (f^{nn_0})_* R = \tau$$

for all continuous currents R in $\mathscr{C}_{k-s}(D_\tau)$. Here, D_τ is the trapping region associated to τ and f^{n_0} by Theorem 1.3.16.

Observe that a similar statement doesn't hold for attracting currents which are not of maximal bidimension in A. Actually, perturbations of a dissipative Hénon map with a robust homoclinic tangency give rise to attracting sets of dimension 1 with infinitely many attracting cycles and thus infinitely many attracting currents of bidimension $(0, 0)$.

The convergence in Theorem 1.3.18 still holds even if R is neither closed nor positive. However, I have not been able to weaken the (unnatural) continuity assumption on R. The equidistributions (1.6) towards the Green currents are conjectured to hold for generic currents but it is known that there exist exceptional currents, i.e. currents $S \in \mathscr{C}_p(\mathbb{P}^k)$ such that

$$\frac{1}{d^p} f^* S = S, \quad \text{and} \quad S \neq T^p.$$

Hence, in particular $d^{-pn}(f^n)^* S \not\to T^p$. My feeling about the case of attracting currents is that there is no exceptional current and the convergence in Theorem 1.3.18 should hold for all $R \in \mathscr{C}_{k-s}(D_\tau)$.

A key ingredient in the proofs of these two theorems is to consider positive closed currents as geometric objects and to use them in order to build new trapping regions. Indeed, the construction we use mimics the one in Section 1.2 given by (1.1). A ϵ-pseudo-orbit of currents of bidimension (s, s) is a sequence $(S_i)_{0\leq i\leq n}$ such that there exist automorphisms $(\sigma_i)_{0\leq i\leq n-1}$ of $\mathbb{P}^k$ which are ϵ-close to the identity and such that

$$\sigma_{i*}\left(\frac{1}{d^s} f_* S_i\right) = S_{i+1}. \tag{1.7}$$

As in (1.1), the union $\mathcal{N}_S(\epsilon)$ of the supports of all ϵ-pseudo-orbits starting at a given current S is a trapping region and the weak $(k-s)$-pseudoconvexity of this region gives tools to study the dynamics on the currents supported in $\mathcal{N}_S(\epsilon)$ (see Section 1.3.8 for more details).

A consequence of the proof of Theorem 1.3.16 is that attracting currents are extremal points in the cone of invariant currents. The proof of Theorem 1.3.18 gives the stronger conclusion that, using the notations of the theorem, the attracting currents of f^{n_0} supported on A are extremal points in the set of currents $\mathscr{D}_{k-s}(A)$ defined as all the possible limit values of

$$\frac{1}{d^{sn}} f^n_* S_n,$$

where $(S_n)_{n\geq 1}$ is a sequence in $\mathscr{C}_{k-s}(A)$. As a byproduct, we obtain the following result.

Corollary 1.3.19. *The support of an attracting current has finitely many connected components. In particular, a minimal quasi-attractor has finitely many connected components.*

There are other relations between the topology of quasi-attractors and their attracting currents but they are less neat. From my opinion, their study deserves to be thorough.

My original motivation behind Theorem 1.3.16 and Theorem 1.3.18 was to prove that every quasi-attractor is actually an attracting set, i.e. every decreasing sequence $(A_i)_{i\geq 1}$ of attracting sets is eventually stationary. This would have greatly simplified their study. However, as shown by Theorem 1.6.8, there exist quasi-attractors in $\mathbb{P}^2$ which are not attracting sets. Nevertheless, a consequence of the finiteness in Theorem 1.3.18 is that, from the point of view of currents, attracting sets and quasi-attractors are the same.

Corollary 1.3.20. *If K is equal to the intersection of a decreasing sequence of attracting sets $(A_i)_{i\geq 0}$ of dimension s then there exists an integer $i_0 \geq 0$ such that the attracting currents of A_i are equal to those of A_{i_0} for all $i \geq i_0$. Moreover, the minimal elements in the set of dimension s quasi-attractors are in one-to-one correspondence with the set of attracting currents in $\mathscr{C}_{k-s}(\mathbb{P}^k)$. In particular, a holomorphic endomorphism of $\mathbb{P}^k$ admits at most countably many minimal quasi-attractors.*

It is classical (see, e.g., [FS01]) that the intersection of an invariant current with the appropriate Green current gives an invariant measure. Hence, to an attracting current τ of bidimension (s, s) is associated an *equilibrium measure* ν_τ defined by

$$\nu_\tau := \tau \wedge T^s.$$

Using standard techniques ([BS92], [dT08], [Din07]) it is easy to deduce from Theorem 1.3.16 and Theorem 1.3.18 the following properties for ν_τ.

Corollary 1.3.21. *Let A be a quasi-attractor of dimension s for f. If τ is an attracting current in $\mathscr{C}_{k-s}(A)$ then its equilibrium measure ν_τ is an ergodic measure of maximal entropy $s \log d$ on A and it has at least s positive Lyapunov exponents. Moreover, if n_0 is the integer defined in Theorem 1.3.18 then ν_τ has at most n_0 ergodic components with respect to f^{n_0}, each of which is mixing.*

In particular, if the chain recurrent set of f is not the union of the class $[\mathscr{J}_k]$ and of attracting cycles then there exists a measure ν supported on $\mathscr{J}_1 \setminus \mathscr{J}_k$ with positive entropy and which is mixing (after replacing f by an iterate). Notice that Theorem 1.3.1 is the combination of Corollary 1.3.19 and Corollary 1.3.21.

Unfortunately, I was not able to prove that ν_τ is always hyperbolic. It is the case in all known examples cf. [Taf13], [DT18a]. If the convergence in Theorem 1.3.18 has exponential speed then the hyperbolicity of ν_τ would probably follow from an adaptation of arguments in [dT08], at least on $\mathbb{P}^2$. The property of ν_τ to be hyperbolic would have several consequences, in particular when $k = 2$ thanks to this result of Daurat.

Theorem 1.3.22 ([Dau18]). *Let $A \subset \mathbb{P}^2$ be an attracting set of dimension 1 ans let $\tau \in \mathscr{C}_1(A)$ be an attracting current. If the measure ν_τ is hyperbolic of saddle type then*

- *the Green current T is laminar in a neighborhood of* $\mathrm{supp}(\nu_\tau)$ *with a laminar structure subordinate to the stable manifolds of ν_τ,*
- *for each $n \geq 1$ there exists a set of saddle periodic points $\mathcal{P}_n$ such that*

$$\nu_\tau = \lim_{n\to\infty} \frac{1}{d^n} \sum_{p\in\mathcal{P}_n} \delta_p.$$

Moreover, it seems likely that $\mathcal{P}_n \subset \mathrm{supp}(\nu_\tau)$. Hence, if these equilibrium measures are always hyperbolic there should not exist quasi-attractors without periodic points in $\mathbb{P}^2$. Notice that [dTN15] adapted the *closing lemma* of Katok to our setting. It implies that the support of a hyperbolic measure is always contained in the closure of the set of periodic points. However, with this approach, it seems difficult to prove that some periodic points are exactly in the support.

Finally, the hyperbolicity of these measures may help to study the *homoclinic classes* which are totally unexplored objects in complex dynamics, except for polynomial automorphisms of $\mathbb{C}^2$ where the situation is simple (there exists a unique homoclinic class).

Remark 1.3.23. *Polynomial automorphisms of $\mathbb{C}^2$ have been extensively studied (see e.g., [BS91a] and [BLS93]). In particular, one can associate to such a map filled Julia sets K^+, K^-, $K = K^+ \cap K^-$ and Julia sets $J^\pm = \partial K^\pm$, $J = J^+ \cap J^-$. Moreover, there exist two positive closed $(1,1)$-currents μ^+ and μ^- with* $\mathrm{supp}(\mu^\pm) = J^\pm$. *The measure $\mu := \mu^+ \wedge \mu^-$ is mixing, hyperbolic, the saddle periodic points equidistribute towards it, its support is contained in J and is the unique homoclinic class and μ is the unique measure of maximal entropy* $\log d$. *It is easy to see that a generic small perturbation of such automorphisms gives an endomorphism f of $\mathbb{P}^2$ which possesses an attracting set A. The map f has its Green current T whose support is exactly the Julia set J_f and there exists a unique attracting current τ supported in A. There is a strong analogy between these objects. The currents T and τ correspond respectively to μ^+ and μ^-, the sets J^+ and K^- correspond to J_f and A. Moreover, in this situation the combination of the results of [Din07], [Dau18], Theorem 1.3.22 and Theorem 1.4.4 gives that the measure $\nu := T \wedge \tau$ is mixing, hyperbolic, the saddle periodic points equidistribute towards it and it is the unique measure supported in A of maximal entropy* $\log d$ *i.e. it can be seen as the continuation of μ for f.*

I will now give the main ideas of the proofs of these results. The first step concernes the dimension of a quasi-attractor.

1.3.5 Dimension of a quasi-attractor and attracting currents

To simplify the notation, for the rest of this chapter s and p will always be integers such that $s + p = k$.

The key point in the proof of Proposition 1.3.15 is that the action of f_* on $H^{p,p}(\mathbb{P}^k, \mathbb{R})$, which is multiplication by d^s, dominates the action on $H^{p+1,p+1}(\mathbb{P}^k, \mathbb{R})$, which is multiplication by d^{s-1}. This gives a way to construct positive *closed* currents supported in a trapping region U from positive currents, *not necessarily closed*, also supported on U. Let us explain this construction for smooth currents. It will illustrate the role of the Green current T^s and of its support $\mathscr{J}_s$.

Let $S \in \mathscr{C}_p(\mathbb{P}^k)$ be a smooth form and let $\chi \geq 0$ be a smooth function. The mass of the push-forwards $d^{-sn} f^n_*(\chi S)$ is

$$\left\langle \frac{1}{d^{sn}} f^n_*(\chi S), \omega^s \right\rangle = \left\langle \chi S, \frac{1}{d^{sn}} f^{n*} \omega^s \right\rangle.$$

Since χS is smooth, (1.6) implies that this last terme converges to $c := \langle T^s, \chi S \rangle$. In particular, if $c \neq 0$ then $\|f^n_*(\chi S)\| \sim c d^{sn}$. On the other hand, $dd^c(\chi S)$ is a smooth $(p+1, p+1)$-form so there exists $M > 0$ such that $-M\omega^{p+1} \leq dd^c(\chi S) \leq M\omega^{p+1}$. The same argument as above shows that $\|f^n_* \omega^{p+1}\| \sim d^{(s-1)n}$. Thus $d^{-sn} f^n_*(dd^c \chi S)$ converges to 0 and every limit value of $d^{-sn} f^n_*(\chi S)$ is dd^c-closed. Actually, the smoothness assumption on S is unnecessary and, using the self-map F of $\mathbb{P}^k \times \mathbb{P}^k$ defined by $F(x, y) = (f(x), f(y))$, Dinh proved that the limit values are d-closed [Din07, Proposition 4.7] (see also [FS98, Proposition 5.4]). Hence we have

Lemma 1.3.24. *Let χ be a positive smooth function in $\mathbb{P}^k$. If S is a current in $\mathscr{C}_p(\mathbb{P}^k)$ then the sequence $d^{-sn}(f^n)_*(\chi S)$ has bounded mass and each of its limit values is a positive closed (p, p)-current of $\mathbb{P}^k$ of mass $c := \langle S \wedge T^s, \chi \rangle$.*

The following result follows then easily and it implies directly Proposition 1.3.15

Proposition 1.3.25. *Let $A \subset \mathbb{P}^k$ be an attracting set for f. For $0 \leq s, p \leq k$ with $s + p = k$, the following properties are equivalent.*

(1) $A \cap \mathscr{J}_s \neq \varnothing$,

(2) $\mathscr{C}_p(A) \neq \varnothing$,

(3) $\mathscr{C}_s(\mathbb{P}^k \setminus A) = \varnothing$.

Proof. Let U be a trapping region associated to A. If $A \cap \mathscr{J}_s \neq \varnothing$ then there exists a smooth function $\chi \geq 0$ supported on U such that $\langle T^s, \chi\omega^p \rangle =: c \neq 0$. By Lemma 1.3.24, the limit values of $\frac{1}{cd^{sn}}(f^n)_*(\chi\omega^p)$ are in $\mathscr{C}_p(\mathbb{P}^k)$ and, since $\overline{f(U)} \subset U$ and $A = \cap_{n \geq 0} f^n(U)$, they must be supported on A. This give (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (3) simply follows from the fact that, for cohomological reasons, the supports of $S \in \mathscr{C}_p(\mathbb{P}^k)$ and $R \in \mathscr{C}_s(\mathbb{P}^k)$ must intersect.

Finally, if $A \cap \mathscr{J}_s = \varnothing$ then $T^s \in \mathscr{C}_s(\mathbb{P}^k \setminus A)$ and thus $\mathscr{C}_s(\mathbb{P}^k \setminus A) \neq \varnothing$. □

As we already said, $\mathscr{J}_k$ is the support of the unique measure $\mu = T^k$ of maximal entropy $k \log d$. The other Julia sets are also related to the topological entropy of f restricted to compact subsets of $\mathbb{P}^k$ (see [Din07] and [dT06]).

Theorem 1.3.26. *If $K \subset \mathbb{P}^k$ is a compact set such that $K \cap \mathscr{J}_{s+1} = \varnothing$ then the topological entropy of f restricted to K is smaller than or equal to $s \log d$.*

In particular, if K is a quasi-attractor of dimension s then its topological entropy is smaller than or equal to $s \log d$. The existence of a measure on K with entropy $s \log d$ will show that K has dimension s if and only if its entropy is exactly $s \log d$.

1.3.6 Structural varieties and geometry of the space of currents $\mathscr{C}_p(U)$

A crucial consequence of Proposition 1.3.15 is that a quasi-attractor K of dimension s verifies the weak p-pseudoconvexity condition ($p + s = k$) introduced in [DS09].

Definition 1.3.27. *A compact subset K of a complex manifold X of dimension k is weakly p-pseudoconvex if there exists a positive smooth (s,s)-form ϕ such that $dd^c\phi$ is strictly positive on K.*

In particular, if there is a current $S \in \mathscr{C}_{s+1}(\mathbb{P}^k \setminus K)$ then there exists a (s,s)-current ϕ such that $dd^c\phi = \omega^{s+1} - S$ and so $dd^c\phi$ is strictly positive on K. Using a regularization and adding a large constant times ω^s, we can assume that ϕ is smooth and positive, thus K is weakly p-pseudoconvex (see [DS09]).

An important point about weakly p-pseudoconvex compact sets in $\mathbb{P}^k$ is that a current of bidegree (p,p) supported on such a set is totally determined by its values on smooth forms ϕ with $dd^c\phi \geq 0$.

Lemma 1.3.28. *Let $U \subset \mathbb{P}^k$ be an open set such that $\overline{U}$ is weakly p-pseudoconvex. Let $R, S \in \mathscr{C}_p(U)$. If for every smooth positive (s,s)-form ϕ on U with $dd^c\phi \geq 0$ there currents verify*

$$\langle R, \phi\rangle = \langle S, \phi\rangle$$

then $R = S$.

The fact that it is enough to test a current on forms ϕ with $dd^c\phi \geq 0$ becomes crucial when combined with the following objects.

A structural variety of positive closed (p,p)-currents parametrized by a complex manifold M is a family of currents $(S_\theta)_{\theta\in M}$ in $\mathscr{C}_p(\mathbb{P}^k)$ such that the currents S_θ "glue" together in order to form a closed current in $M \times \mathbb{P}^k$. To be more precise, there exists a positive closed (p,p)-current $\mathcal{S}$ on $M \times \mathbb{P}^k$ such that for every $\theta \in M$ the *slice* at θ of $\mathcal{S}$ is equal to S_θ. We refer to [Fed69] for the slicing theory of currents of Federer. Structural varieties played an important role in the study of horizontal maps (see [Duj04], [DS06], [DNS08]) and Dinh and Sibony have systematized their use in the theory of *super-potential* (see [DS09], [DS10c]). In particular, these authors showed that the function $\theta \mapsto \langle S_\theta, \phi\rangle$ inherits properties from the test form ϕ.

Theorem 1.3.29. *[DS06, Theorem 2.1][Din07, Proposition A.1] Let U be an open subset of $\mathbb{P}^k$ such that $\overline{U}$ is weakly p-pseudoconvex. If $(S_\theta)_{\theta\in M}$ is a structural variety in $\mathscr{C}_p(U)$ and if ϕ is a real continuous (s,s)-form on U such that $dd^c\phi \geq 0$ (resp. $dd^c\phi = 0$, resp. $d\phi = 0$) then the function $h(\theta) := \langle S_\theta, \phi\rangle$ is plurisubharmonic (resp. pluriharmonic, resp. constant). In particular, the mass of S_θ is independent of θ.*

In particular, if $\overline{U}$ is weakly p-pseudoconvex then the structural varieties in $\mathscr{C}_p(U)$ can be fully studied by the way of plurisubharmonic functions.

All the structural varieties that we consider in what follows can be obtained from this family of examples using convex combinations and limits.

Example 1.3.30. *Let $U \subset \mathbb{P}^k$ be as in Theorem 1.3.29 and let $S \in \mathscr{C}_p(U)$. For a holomorphic map $g\colon M \times U \to U$, denote by $g_\theta(z) := g(\theta, z)$. The number $d_g := \|(g_\theta)_*S\|$ is independent of θ and if $d_g \neq 0$ then*

$$S_\theta := \frac{1}{d_g}(g_\theta)_*S$$

defined a structural variety parametrized by M in $\mathscr{C}_p(U)$.

The structural varieties will be used mainly in two ways and in both cases, it will be sufficient to consider *structural disks,* i.e. structural varieties parametrized by the unit disk $\mathbb{D}$ of $\mathbb{C}$. If $S \in \mathscr{C}_p(U)$ then there exist structural varieties

- $(S_\theta)_{\theta\in\mathbb{D}}$ regularizing S, i.e. $S_0 = S$ and S_θ is smooth if $\theta \in \mathbb{D}^*$,
- and other ones $(R_\theta)_{\theta\in\mathbb{D}}$ which corresponds to pseudo-orbits of length n, i.e. $R_0 = d^{-sn} f_*^n S$ and R_θ is the last term of a ϵ-pseudo-orbit of S as in (1.7), where ϵ depends on θ.

We refer to Section 1.3.7 and Section 1.3.8 for more details.

In [Din07], Dinh studies attracting sets associated to trapping regions with the following properties.

There exist two linear subspaces I and L of dimension $p-1$ and s respectively such that $I \cap U = \varnothing$ and $L \subset U$. Moreover, for each $x \in L$ the unique dimension p linear subspace $I(x)$ containing I and x intersects U in a subset which is star-shaped with respect to x in $I(x) \setminus I \simeq \mathbb{C}^p$. (HD)

A typical example of an attracting set with a trapping region U satisfying (HD) is the hyperplane at infinity H_∞ in $\mathbb{P}^k = \mathbb{C}^k \cup H_\infty$ when f is a polynomial endomorphism of $\mathbb{C}^k$ which extends holomorphically to $\mathbb{P}^k$. Observe that these assumptions are only about trapping regions and that small perturbations of f still admit U as a trapping region. In this example, the dimension of the attracting set is $k-1$ but similar examples of any dimension $0 \leq s \leq k$ are easy to obtain (see Example 1.6.1).

The main two consequences of (HD) are the following. If U satisfies (HD) then

- $[I] \in \mathscr{C}_{s+1}(\mathbb{P}^k \setminus U)$ thus all the compact subsets of U are weakly p-pseudoconvex,
- the set of currents $\mathscr{C}_p(U)$ is connected by structural disks.

Let's give some precisions on this last point. There exist homogeneous coordinates $[x_0 : \cdots : x_k]$ of $\mathbb{P}^k$ such that $I = \{x_0 = \cdots = x_s = 0\}$ and $L = \{x_{s+1} = \cdots = x_k = 0\}$. For each $\theta \in \mathbb{D}^*$ the self-map ρ_θ of $\mathbb{P}^k$ defined by

$$\rho_\theta([x_0 : \cdots : x_k]) = [x_0 : \cdots : x_s : 2\theta x_{s+1} : \cdots : 2\theta x_k]$$

is an automorphism such that $\rho_{1/2} = \mathrm{Id}$. The corresponding map for $\theta = 0$ is the projection of $\mathbb{P}^k \setminus I$ onto L. Therefore, if $S \in \mathscr{C}_p(U)$ then

$$S_\theta := (\rho_\theta)_* S$$

is a structural disk such that $S_{1/2} = S$ and $S_0 = [L]$. Moreover, the star-shaped assumption in (HD) implies that this disk restricted to the parameters $|\theta| < r$, with $r > 1/2$ small enough, is a structural disk in $\mathscr{C}_p(U)$. In some sense, $\mathscr{C}_p(U)$ is star-shaped with respect to $[L]$.

The following simple example shows that the setting is radically different in general.

Example 1.3.31. *Let f be the endomorphism of $\mathbb{P}^2$ defined by $f([x_0 : x_1 : x_2]) = [x_1^2 : x_0^2 : x_3^2]$. For $i \in \{1, 2, 3\}$, define $H_i = \{x_i = 0\}$ and let U_i be a small neighborhood of H_i. These neighborhoods can be chosen such that each of them satisfies* (HD) *and $U := U_0 \cup U_1$ is a trapping region for f. Then $\mathscr{C}_p(U)$ has uncountably many connected components with respect to structural disks. To be more precise, for $t \in [0,1]$ define $R_t := t[H_1] + (1-t)[H_0]$. If $t_1 \neq t_2$ then R_{t_1} and R_{t_2} cannot be joined by a chain of structural disks. Actually, if $S \in \mathscr{C}_1(U_2)$ is smooth and χ is a smooth function such that $\chi = 1$ on $U_1 \cap U_2$ and $\chi = 0$ on U_0 then*

- $\langle R_t, \chi S\rangle = t$ *and*
- *by Theorem 1.3.29,* $\theta \mapsto \langle S_\theta, \chi S\rangle$ *is constant for every structural disk* $(S_\theta)_{\theta\in\mathbb{D}}$ *in* $\mathscr{C}_1(U)$ *since* χS *is closed on* U.

The same situation happens with a small neighborhood V an attracting cycle of period 2 in $\mathbb{P}^1$ but we choose this one in $\mathbb{P}^2$ to emphasized that the problem doesn't come from the lake of connectedness of V. Moreover, $\mathscr{C}_1(V)$ has only two connected components (with respect to structural disks) containing extremal currents while $\mathscr{C}_1(U)$ has infinitely many (consider the normalized current of integration on $X_{a,b} = \{x_0^a x_1^b = \epsilon\}$, $a, b \in \mathbb{N}$, which is in a structural disk with R_t, $t = b/(a + b)$).

I finish this section with two remarks about attempts (unsuccessful until now) to put nice additional structures on $\mathscr{C}_p(U)$ in order to have a better understanding of it. Both these ideas underlie most of the proofs of [Taf18].

The first one is about defining a good distance on $\mathscr{C}_p(U)$. The hope is to find conditions on U which ensure that the iterates f^n act equicontinuously on $\mathscr{C}_p(U)$, i.e. it can be considered as a Fatou component in the set of currents.

Remark 1.3.32. *In [DS06] Dinh and Sibony defined a Kobayashi pseudo-distance on* $\mathscr{C}_p(U)$ *mimicking the one on complex manifold. The pseudo-distance between* $S, R \in \mathscr{C}_p(U)$ *is the infimum of* $\sum_{n=1}^N \mathrm{dist}_{\mathbb{D}}(a_n, b_n)$ *over all the possible chains of structural disks* $(S_{n,\theta})_{\theta\in\mathbb{D}}$ *in* $\mathscr{C}_p(U)$, $1 \le n \le N$, *with*

$$S = S_{1,a_1}, \quad S_{n,b_n} = S_{n+1,a_{n+1}} \quad \text{and} \quad S_{N,b_N} = R.$$

Here, $\mathrm{dist}_{\mathbb{D}}$ *is the Poincaré distance on* $\mathbb{D}$. *If this pseudo-distance is a distance, it is natural to say that* $\mathscr{C}_p(U)$ *is Kobayashi hyperbolic. They prove that if* $\overline{U}$ *is weakly* p*-pseudoconvex then* $\mathscr{C}_p(\overline{U})$ *is Brody hyperbolic, i.e. there is no non-constant structural variety in* $\mathscr{C}_p(\overline{U})$ *which is parametrized by* $\mathbb{C}$. *However, they also show that even in the simple case of measure on* $\mathbb{D}$, $\mathscr{C}_1(\mathbb{D})$, *this pseudo-distance is not a distance. Actually, they show that the distance between the Dirac mass at* 0 *and the Lebesgue measure on a circle centered at* 0 *vanishes. The structural disks they use to this end are similar to the ones in Example 1.3.30 except that the map* g *is not holomorphic and given by*

$$g(\theta, z) = |\theta||z|^{1/n},$$

for $n \ge 1$. *Dinh-Sibony present structural varieties as analogs to complex subvarieties in the infinite dimensional space* $\mathscr{C}_p(U)$. *However, these varieties are rather of plurisubharmonic nature and they seem too flexible to define a good notion of Kobayashi pseudo-distance. When* U *is a trapping region, it should be possible to find a subfamily of structural disks, stable by the dynamics, such that the Kobayashi pseudo-distance defined with these disks is a distance on* $\mathscr{C}_p(U)$.

The pre-order defined in the following remark uses structural disk to compare the supports of currents. In some sense, the support of a current which is minimal for this relation can be considered as "irreducible". This idea, restricted to invariant currents, is a key ingredient in the proof of Theorem 1.3.18.

Remark 1.3.33. *Let* $R, S \in \mathscr{C}_p(U)$. *One can say that* $R \vdash S$ *if there exists a structural disk* $(R_\theta)_{\theta\in\mathbb{D}}$ *in* $\mathscr{C}_p(U)$ *such that* $R = R_0$ *and* $\mathrm{supp}(S) \subset \mathrm{supp}(R_\theta)$ *for some* $\theta \in \mathbb{D}$. *Since we work on* $\mathbb{P}^k$, *a equivalent formulation is that there exists a second structural disk* $(S_\theta)_{\theta\in\mathbb{D}}$ *such that* $R_\theta \ge cS_{\theta'}$ *for some* $\theta, \theta' \in \mathbb{D}$ *and* $c > 0$. *If we extend this relation in order to force transitivity then* $\vdash$ *becomes a pre-order. For measures, i.e.* $p = k$, *the minimal elements in* $\mathscr{C}_k(U)$ *corresponds to connected components of* U. *If* U *is as in Example 1.3.31 then* $\mathscr{C}_1(U)$ *has* 2 *minimal classes corresponding to* $[H_0]$ *and* $[H_1]$.

1.3.7 Existence and uniqueness of attracting currents under (HD)

In this section we sketch the proof of the main result of [Din07]. We slightly modify one of the arguments of the original proof in order to be still able to use it later in our more general setting.

Let U be a trapping region satisfying (HD) and let $A = \cap_{n\geq 0} f^n(U)$ be the associated attracting set. As we have seen in Section 1.3.6, the assumptions (HD) on U imply that if $S \in \mathscr{C}_p(U)$ then there exists a structural disk $(S_\theta)_{\theta\in\mathbb{D}}$ in $\mathscr{C}_p(U)$ such that $S_{1/2} = S$ and $S_0 = [L]$. Using convolution, Dinh obtained a disk $(R_\theta)_{\theta\in\mathbb{D}}$ such that

(1) $R_{1/2} = S$,

(2) if $\theta \neq 1/2$, then R_θ is a continuous form,

(3) for each compact subset L of $\mathbb{D} \setminus \{1/2\}$ there exists a constant $c_L > 0$, independent of S, such that $\|R_\theta - R_{\theta'}\|_{\mathcal{C}^0} \leq c_L|\theta - \theta'|$, for all $\theta, \theta' \in L$,

(4) R_0 is independent of S.

The only point that requires (HD) is the last one. We modify it to this more flexible one:

(4') there exist $c > 0$ and $\widetilde{R} \in \mathscr{C}_p(U)$ which is strictly positive on a neighborhood of A such that $R_0 \geq c\widetilde{R}$. Both $\widetilde{R}$ and c are independent of S.

From this, the proof goes as follows.

Step 0: The set $\mathscr{D}_p(U)$.

In order to simplify the notation, denote

$$\Lambda := d^{-s} f_*$$

the push-forward operator associated to f. The set of currents $\mathscr{D}_p(U)$ is defined as all the possible limit values of sequences of the form

$$\Lambda^n S_n$$

where $(S_n)_{n\geq 0}$ is a sequence in $\mathscr{C}_p(U)$. In particular, $\mathscr{D}_p(U)$ contains all the limit values of $\Lambda^n S$ with $S \in \mathscr{C}_p(U)$. Moreover, $\mathscr{D}_p(U)$ is compact, all its elements are supported in A and it is easy to see that $S \in \mathscr{D}_p(U)$ if and only if there exists $S_n \in \mathscr{D}_p(U)$ such that $S = \Lambda^n S_n$.

Step 1: Construction of the attracting current τ.

The set $\overline{U}$ is weakly p-pseudoconvex thus the currents in $\mathscr{C}_p(U)$ are entirely determined by their values on smooth forms $\phi \geq 0$ with $dd^c\phi \geq 0$. Let ϕ be such a form and define

$$c_\phi := \sup_{S\in\mathscr{D}_p(U)} \langle S, \phi\rangle.$$

As $\mathscr{D}_p(U)$ is compact, there exists $S \in \mathscr{D}_p(U)$ with $\langle S, \phi\rangle = c_\phi$ and as we have said, there exists $S_n \in \mathscr{D}_p(U)$ such that $S = \Lambda^n(S_n)$. Let $(R_{n,\theta})_{\theta\in\mathbb{D}}$ be the structural disk associated to S_n satisfying the points **(1)-(4')**. By Theorem 1.3.29, the functions

$$h_n(\theta) = \langle \Lambda^n(R_{n,\theta}), \phi\rangle$$

are subharmonic. If $\theta \in \mathbb{D}$ then $\limsup_{n\to\infty} h_n(\theta) \leq c_\phi$ since all possible limit values of $\Lambda^n(R_{n,\theta})$ are in $\mathscr{D}_p(U)$. Moreover, $h_n(0) = c_\phi$ and the point **(3)** above implies that the functions h_n are locally equicontinuous on $\mathbb{D} \setminus \{1/2\}$. Therefore, by Lemma 1.3.7 the sequence h_n converges pointwise to c_ϕ. In particular,

$$\lim_{n\to\infty} \langle \Lambda^n(R_{n,0}), \phi\rangle = c_\phi.$$

On the other hand, $R_{n,0} \geq c\widetilde{R}$ thus

$$\limsup_{n\to\infty} \langle \Lambda^n(R_{n,0} - c\widetilde{R}), \phi\rangle \leq (1-c)c_\phi \quad \text{and} \quad \limsup_{n\to\infty} \langle \Lambda^n \widetilde{R}, \phi\rangle \leq c_\phi.$$

Hence, Lemma 1.3.8 gives that $\langle \Lambda^n \widetilde{R}, \phi\rangle$ converges to c_ϕ. Since $\widetilde{R}$ is independent of ϕ this implies that $\Lambda^n \widetilde{R}$ converges to a current τ such that, for all ϕ with $dd^c\phi \geq 0$, $\langle \tau, \phi\rangle = c_\phi$.

Step 2: Convergence towards τ.

By assumption $\widetilde{R}$ is strictly positive on a neighborhood V of A. Hence, for each continuous form $S \in \mathscr{C}_p(V)$ there is $c > 0$ such that $\widetilde{R} \geq cR$. The same arguments than above then imply that, for all ϕ with $dd^c\phi \geq 0$, $\langle \Lambda^n R, \phi\rangle$ converges to c_ϕ, i.e. $\Lambda^n R$ converges to τ. From this, it is not difficult to extend the convergence to all continuous form in $\mathscr{C}_p(U)$.

In particular, as the convergence holds for all continuous form in $\mathscr{C}_p(U)$, the current τ is automatically the unique attracting current in $\mathscr{C}_p(U)$.

1.3.8 Dynamics in the space of currents

There are several major differences between an arbitrary trapping region and one satisfying (HD). The first one is that, as shown by the simple example of an attracting 2-cycle, a sequence of the form $\Lambda^n R$ may not converge in general. Hence, instead of the operators Λ^n, we consider their Cesàro means

$$\Delta_n := \frac{1}{n}\sum_{i=1}^{n} \Lambda^i.$$

All the limit values of $\Delta_n S$ are in the set $\mathscr{I}_p(U)$ of elements of $\mathscr{C}_p(U)$ invariant by Λ. Hence, we restrict ourselves, at least at the beginning, to this set instead of $\mathscr{D}_p(U)$.

Another important difficulties is that there might be several attracting currents and thus they cannot simply be defined by the formula

$$\langle \tau, \phi\rangle := \max \langle S, \phi\rangle,$$

where the max is over $\mathscr{D}_p(U)$ or $\mathscr{I}_p(U)$. To solve this problem, we keep the idea that an attracting current must maximise test forms ϕ with $dd^c\phi \geq 0$ but not all at the same time. To be more precise, let $(\phi_n)_{n\geq 1}$ be a dense sequence in the space of test forms ϕ with $dd^c\phi \geq 0$. To this sequence we will associated an attracting current and the construction will depend on the order in the sequence. This construction implies Theorem 1.3.16 and here are its main steps.

Step 1: Trapping regions associated to invariant currents.

By compactness of $\mathscr{I}_p(U)$, there exists $S_1 \in \mathscr{I}_p(U)$ such that

$$\langle S_1, \phi_1\rangle = \max_{S\in\mathscr{I}_p(U)} \langle S, \phi_1\rangle.$$

Now, we will associate a trapping region to S_1. A first important remark is that, since U is a trapping region for f, there exists an open neighborhood W of the identity in

$\mathrm{Aut}(\mathbb{P}^k)$ such that for every $\sigma \in W$, $\sigma \circ f(U) \subset U$. From this, we say that $(R_i)_{0\leq i\leq n}$ is a *W-pseudo-orbit* between S and R if there exists $\{\sigma_0, \dots, \sigma_{n-1}\} \subset W$ such that

$$S = R_0, \quad R_{i+1} = \sigma_{i*}(\Lambda R_i) \quad \text{and} \quad R_n = R,$$

or equivalently

$$R = (\sigma_{n-1})_* \Lambda \cdots (\sigma_0)_* \Lambda S. \tag{1.8}$$

Define $\mathcal{N}_{S_1}$ by

$$\mathcal{N}_{S_1} = \cup_R \mathrm{supp}(R),$$

where the union is over all the currents R such that there exists a W-pseudo-orbit between S_1 and R. The properties of W and the invariance of S_1 give easily

- $S_1 \in \mathscr{C}_p(\mathcal{N}_{S_1})$,
- $\mathcal{N}_{S_1}$ is an open subset of U,
- for every $\sigma \in W$, $\sigma \circ f(\mathcal{N}_{S_1}) \subset \mathcal{N}_{S_1}$ and in particular $\overline{f(\mathcal{N}_{S_1})} \subset \mathcal{N}_{S_1}$.

Step 2: Structural disks and convergence in $\mathcal{N}_{S_1}$.

Possibly by reducing W, we can assume that there exists a biholomorphism ψ between W and a unit ball $\mathbb{B}$ in some $\mathbb{C}^N$. This allows us to define, for $\sigma \in W$ and $\theta \in \mathbb{D}$, the product $\theta\sigma := \psi^{-1}(\theta\psi(\sigma))$. From this, we can retract a W-pseudo-orbit between two currents S and R to an authentic orbit of S using a structural disk. Actually, it is enough to deform (1.8) by

$$R_\theta := (\theta\sigma_{n-1})_* \Lambda \cdots (\theta\sigma_0)_* \Lambda S.$$

This structural disk (parametrized by a slightly larger disk than $\mathbb{D}$) verifies $R_1 = R$ and $R_0 = \Lambda^n S$. In particular, if S is invariant by Λ then $R_0 = S$. Using this kind of disks and a regularization process, it is easy to show that there exists a structural disk $(R_\theta)_{\theta\in\mathbb{D}}$ in $\mathscr{C}_p(\mathcal{N}_{S_1})$ such that $R_0 = S_1$ and for every continuous form $R \in \mathscr{C}_p(\mathcal{N}_{S_1})$ there are $c > 0$ and $\theta_0 \in \mathbb{D}$ such that

$$R_{\theta_0} \geq cR.$$

Using similar arguments to those of Section 1.3.7 we then obtain that for every continuous form $R \in \mathscr{C}_p(\mathcal{N}_{S_1})$ each limit value R_∞ of $(\Delta_l R)_{l\geq 1}$ must satisfies

$$\langle R_\infty, \phi_1 \rangle = \langle S_1, \phi_1 \rangle.$$

Step 3: Induction.

To summarize the first step, we started with a trapping region U and an open set W of $\mathrm{Aut}(\mathbb{P}^k)$ such that

$$\forall \sigma \in W, \quad \sigma \circ f(U) \subset U.$$

And we obtained a trapping region $\mathcal{N}_{S_1} \subset U$ which satisfies

$$\forall \sigma \in W, \quad \sigma \circ f(\mathcal{N}_{S_1}) \subset \mathcal{N}_{S_1},$$

and this without reducing W.

We can now apply the same idea to $\mathcal{N}_{S_1}$ instead of U. Namely, there exists $S_2 \in \mathscr{I}_p(\mathcal{N}_{S_1})$ such that

$$\langle S_2, \phi_2 \rangle = \max_{S \in \mathscr{I}_p(\mathcal{N}_{S_1})} \langle S, \phi_2 \rangle.$$

In the same way, we obtain a trapping region $\mathcal{N}_{S_2} \subset \mathcal{N}_{S_1}$ such that

$$\forall \sigma \in W, \quad \sigma \circ f(\mathcal{N}_{S_2}) \subset \mathcal{N}_{S_2}.$$

Applying **Step 2** to this trapping region implies that for every continuous form $R \in \mathscr{C}_p(\mathcal{N}_{S_2})$ each limit value R_∞ of $(\Delta_l R)_{l\geq 1}$ must satisfies

$$\langle R_\infty, \phi_2 \rangle = \langle S_2, \phi_2 \rangle.$$

But a continuous form in $\mathscr{C}_p(\mathcal{N}_{S_2})$ is also in $\mathscr{C}_p(\mathcal{N}_{S_1})$ so

$$\langle R_\infty, \phi_1 \rangle = \langle S_1, \phi_1 \rangle,$$

i.e. R_∞ is an element of $\mathscr{I}_p(\mathcal{N}_{S_2})$ which maximises both ϕ_1 and ϕ_2. Hence, we can assume that it is also the case for S_2, i.e. $\langle S_2, \phi_1 \rangle = \langle S_1, \phi_1 \rangle$.

By induction, this gives a decreasing sequence of trapping regions $\mathcal{N}_{S_n}$ associated to invariant currents S_n such that

(1) for all $\sigma \in W$, $\sigma \circ f(\mathcal{N}_{S_n}) \subset \mathcal{N}_{S_n}$,

(2) for $1 \leq i \leq n$,
$$\langle S_n, \phi_i \rangle = \max_{S \in \mathscr{I}_p(\mathcal{N}_{S_i})} \langle S, \phi_i \rangle,$$

(3) for all continuous form R in $\mathscr{C}_p(\mathcal{N}_{S_n})$ each limit value R_∞ of $(\Delta_l R)_{l\geq 1}$ satisfies
$$\langle R_\infty, \phi_n \rangle = \langle S_n, \phi_n \rangle.$$

Step 4: The sequence $(\mathcal{N}_{S_n})_{n\geq 1}$ is eventually stationary and conclusion.

The key point in **Step 3** is that all the trapping regions $\mathcal{N}_{S_n}$ satisfies

$$\forall \sigma \in W, \quad \sigma \circ f(\mathcal{N}_{S_n}) \subset \mathcal{N}_{S_n},$$

where W is independent of n. It is easy to see that there can be only finitely many attracting sets associated to a decreasing sequence of trapping regions with this property. In other words, if $A_n := \cap_{i\geq 0} f^i(\mathcal{N}_{S_n})$ then there exists $n_0 \geq 1$ such that $A_n = A_{n_0}$ for all $n \geq n_0$. In this situation, the construction of $\mathcal{N}_{S_n}$ implies that $\mathcal{N}_{S_n} = \mathcal{N}_{S_{n_0}}$. Therefore, the point **(3)** in **Step 3** implies that if R is a continuous form in $\mathscr{C}_p(\mathcal{N}_{S_{n_0}})$ then for each limit value R_∞ of $(\Delta_l R)_{l\geq 1}$ and each $n \geq 1$

$$\langle R_\infty, \phi_n \rangle = \langle S_n, \phi_n \rangle.$$

In particular, it is independent of R and of the choice of the limit value. Using the density of the sequence $(\phi_n)_{n\geq 1}$, this implies that $\Delta_l R$ converges to a current τ which is independent of R, i.e. τ is attracting on $\mathcal{N}_{S_{n_0}}$.

This ends the proof of Theorem 1.3.16. When the sequence $(\phi_n)_{n\geq 0}$ changes the construction may give a different attracting current.

Remark 1.3.34. *In Example 1.3.31 the open set $U = U_0 \cup U_1$ is a trapping region for the map $f[x_0 : x_1 : x_2] = [x_0^d : x_1^d : x_2^d]$. It is artificially the union of two trapping regions U_0 and U_1 whose associated attracting sets are respectively $H_0 = \{x_0 = 0\}$ and $H_1 = \{x_1 = 0\}$. In the construction above, we have $\mathcal{N}_{S_n} = U$ until ϕ_{n_0} distinguishes H_0 from H_1, i.e. $\langle [H_0], \phi_{n_0} \rangle \neq \langle [H_1], \phi_{n_0} \rangle$. Then, for $n \geq n_0$, $\mathcal{N}_{S_n} = U_i$ and $\tau = [H_i]$, where $\langle [H_i], \phi_{n_0} \rangle = \max_{j=0,1} \langle [H_j], \phi_{n_0} \rangle$. A well-chosen permutation of the sequence $(\phi_n)_{n\geq 1}$ gives the other line as an attracting current.*

However, all the resulting trapping regions are still invariant by $\sigma \circ f$ for all $\sigma \in W$. It is possible to deduce from this that the set of attracting currents in $\mathscr{C}_p(U)$ obtained by this method forms a finite set $\{\tau_1, \ldots, \tau_m\}$ and if ϕ is a test form with $dd^c\phi \geq 0$ on U then

$$\max_{1 \leq i \leq m} \langle \tau_i, \phi \rangle = \max_{S \in \mathscr{I}_p(U)} \langle S, \phi \rangle.$$

The major issue to prove Theorem 1.3.18 is that there might exist infinitely many currents which are attracting on trapping regions which are not invariant by $\sigma \circ f$ for some $\sigma \in W$. In [Taf18, Lemma 3.27], which is the key result of the paper, we show that it is not possible: if a current in $\mathscr{C}_p(U)$ is attracting on V then $\sigma \circ f(V) \subset V$ for all $\sigma \in W$. In other words, the "basin of attraction" of an attracting current in $\mathscr{C}_p(U)$ inherits the invariance properties of U. This implies that the number of attracting currents in $\mathscr{C}_p(U)$ is finite, bounded by a constant depending only on W. In particular, this constant is the same for all the iterates of f which implies the following result.

Proposition 1.3.35. *There exists an integer $n_0 \geq 1$ such that if we replace f by f^{n_0} then for all $n \geq 1$ the set of attracting currents for f^n in $\mathscr{C}_p(U)$ is equal to the set of attracting currents for f in $\mathscr{C}_p(U)$.*

The last step towards Theorem 1.3.18 is to remove the Cesàro mean in the convergence. This is done by studying the structure of $\mathscr{D}_p(U)$ using techniques similar to those to prove Theorem 1.3.16. This part heavily relies on Proposition 1.3.35.

From the finiteness of the set of attracting currents supported on an attracting set, it is easy to obtain results about quasi-attractors.

Theorem 1.3.36. *Let K be a minimal element in the set of quasi-attractors of dimension s. There exists an integer $n_0 \geq 1$ such that if we replace f by f^{n_0} then K splits into n_0 quasi-attractors $K = K_1 \cup \cdots \cup K_{n_0}$ such that each K_i is contained in a trapping region U_{K_i} which supports a unique attracting current $\tau_i \in \mathscr{C}_p(K_i)$ with*

$$\lim_{n \to \infty} \frac{1}{d^{ns}} (f^n)_* R = \tau_i$$

for all continuous currents R in $\mathscr{C}_p(U_{K_i})$. In particular, each K_i is minimal in the set of s-dimensional quasi-attractors of f.

1.3.9 Connected components

One unexpected consequence of the results above concernes the connectivity properties of minimal quasi-attractors: such a quasi-attractor must have finitely many connected components. The idea, which is already available for continuous maps, is that a minimal quasi-attractor which contains an invariant set with finitely many connected components (a periodic orbit for example) has at most the same number of components. In our setting, this invariant set will be the support of an attracting current. The fact that it has finitely many connected components follows from the following extremality result which is a consequence of the last part of the proof of Theorem 1.3.18.

Corollary 1.3.37. *Let τ be a current attracting on a trapping region U. If τ is also attracting for all the iterates f^n then for all test forms ϕ with $dd^c\phi \geq 0$ on U we have*

$$\langle \tau, \phi \rangle = \max_{S \in \mathscr{D}_p(U)} \langle S, \phi \rangle.$$

In particular, τ is extremal in the cone $\mathscr{D}_p(U)$.

Notice that the assumption on τ is not really restrictive since, by Proposition 1.3.35, any attracting current is an average of finitely many attracting currents for f^{n_0} with this property. The first part of Corollary 1.3.19 follows easily.

Corollary 1.3.38. *Let τ and U be as in Corollary 1.3.37. Then the support of τ is connected.*

Proof. Let X be a connected component of $\mathrm{supp}(\tau)$ and let $0 \leq \chi \leq 1$ be a smooth function such that $\chi = 1$ on a small neighborhood of X and $\mathrm{supp}(\chi) \cap \mathrm{supp}(\tau) = X$. For $n \geq 0$, define the current

$$S_n := \frac{(\chi \circ f^n)\tau}{\|\chi\tau\|}.$$

It is clear that these currents are positive and they are also of mass 1 since

$$\|S_n\| = \langle S_n, T^s\rangle = \frac{\langle T^s \wedge \tau, \chi \circ f^n\rangle}{\langle T^s \wedge \tau, \chi\rangle} = 1,$$

since $\nu_\tau := T^s \wedge \tau$ is an invariant measure. Moreover,

$$\Lambda^n S_n = \frac{1}{d^{sn}} f^n_*(\chi \circ f^n \tau) = \chi(\Lambda^n \tau) = \chi\tau = S_0.$$

Finally, we claim that each S_n are closed. This implies that S_0 is in $\mathscr{D}_p(U)$. If $S_0 \neq \tau$ (i.e. $\|\chi\tau\| < 1$) then the same holds for

$$R_0 := \frac{(1-\chi)\tau}{\|(1-\chi)\tau\|}$$

and thus

$$\tau = \|\chi\tau\| S_0 + \|(1-\chi)\tau\| R_0$$

which contradicts to fact that, by Corollary 1.3.37, τ is extremal in $\mathscr{D}_p(U)$.

It remains to prove that S_n is closed for each $n \geq 0$. Let $n \geq 0$ and let $x \in \mathrm{supp}(S_n)$. By the definition of S_n, this implies that $x \in \mathrm{supp}(\tau)$ and $x \in \mathrm{supp}(\chi \circ f^n)$. The first point gives, since $\mathrm{supp}(\tau)$ is invariant by f, that $f^n(x) \in \mathrm{supp}(\tau)$ and the latter that $f^n(x) \in \mathrm{supp}(\chi)$, i.e. $f^n(x) \in \mathrm{supp}(\tau) \cap \mathrm{supp}(\chi) = X$. But $\chi = 1$ in a neighborhood of X and f^n is an open mapping thus $\chi \circ f^n = 1$ in a neighborhood of x. Hence, in this neighborhood S_n coincides with τ and thus is closed. □

In order to obtain the second part of Corollary 1.3.19, we prove a slightly stronger result.

Corollary 1.3.39. *Let K be a minimal element in the set of quasi-attractors of dimension s. Then K has finitely many connected components.*

Proof. By Theorem 1.3.36, if we replace f by f^{n_0} then K is the union of n_0 quasi-attractors K_i each of which supports a unique attracting current satisfying the assumption of Corollary 1.3.38. We will show that each K_i is connected and to simplify the notations, we assume that $n_0 = 1$ and $K = K_i$.

Hence, K is a quasi-attractor, minimal in dimension s, which supports an attracting current τ such that $\mathrm{supp}(\tau)$ is connected. By definition, K is the intersection of a decreasing family of attracting sets $(A_i)_{i\geq 1}$. Each attracting set A_i has finitely many connected components that we denote by $A_{i,j}$. Since $\mathrm{supp}(\tau) \subset K \subset A_i$ is connected, for each $i \geq 1$ there exists $j(i)$ such that $\mathrm{supp}(\tau) \subset A_{i,j(i)}$. A first observation is that, since the sequence $(A_i)_{i\geq 1}$ is decreasing, the same holds for $(A_{i,j(i)})_{i\geq 1}$. On the other hand, the invariance of $\mathrm{supp}(\tau)$ implies that $f(A_{i,j(i)}) = A_{i,j(i)}$ and it is easy to deduce from this that $A_{i,j(i)}$ is an attracting set for f. Hence $K' := \cap_{i\geq 1} A_{i,j(i)}$ is a quasi-attractor such that $\mathrm{supp}(\tau) \subset K'$ and thus it has dimension s. Since $K' \subset K$, the minimality of K implies that $K' = K$. Finally, K' is a decreasing intersection of connected compact sets and thus it is also connected. □

1.4 Attracting sets with small topological degree and generalizations

The purpose of this section is to investigate whether additional assumptions on the dynamics on an attracting set A imply additional properties on its attracting currents and reciprocally. This study started with the work of Daurat [Dau14] where she introduced the notion of attracting sets with *small topological degree*. The idea that discrepancies between the dynamical degrees have strong effects on the dynamics has a long history (see e.g. [Gue10] for the global setting of rational maps and [DS03], [Duj04], [DNS08] for more local settings). Roughly speaking, in dimension 2, when the topological degree is larger than the first dynamical degree then the dynamics has strong similarities with the one of holomorphic endomorphisms of $\mathbb{P}^2$. When the first dynamical degree is larger then the dynamics looks like the one of Hénon maps. This last point has been in particular developed in the works of Diller-Dujardin-Guedj [DDG10a, DDG11, DDG10b] for meromorphic maps on compact surfaces and Daurat used their ideas and methods in the semi-local setting of attracting sets [Dau14, Dau18].

Notice that there are differences between the degrees defined on a compact manifold and those defined in a local setting. In the first case the degrees are mainly related to the action on the cohomology and can be computed using smooth objects. This cohomological nature gives strong constraints such as the Khovanskii-Teissier-Gromov log-concavity of these degrees. In a local setting, the definition can change if we consider smooth or singular objects.

1.4.1 Local dynamical degrees

Let f be an endomorphism of $\mathbb{P}^k$ of degree $d \geq 2$. Let $U \subset \mathbb{P}^k$ be a trapping region of dimension s. Following [DS03] and [DNS08], define for $0 \leq l \leq k$ the dynamical degrees d_l and d_l^* on U by

$$d_l := \limsup_{n\to\infty} \left(\int_U (f^{n*}\omega^l) \wedge \omega^{k-l} \right)^{1/n}$$

and

$$d_l^* := \limsup_{n\to\infty} \left(\sup_{S\in\mathscr{C}_l(\mathbb{P}^k)} \|f^{n*}S\|_U \right)^{1/n}.$$

In some sense, d_l^* corresponds to the maximal exponential growth of the volume of $f^{-n}(X)\cap U$ where X is a codimension l complex submanifold. The degree d_l has a similar interpretation but replacing the maximum by an average.

A first easy observation is that by taking $S = \omega^l$ we get $d_l \leq d_l^*$. A second one is that $\|f^{n*}S\|_U \leq \|f^{n*}S\| = d^{ln}$, thus $d_l^* \leq d^l$. Finally, since the dimension of U is s, for all $0 \leq l \leq s$ there exist smooth forms in $\mathscr{C}_{k-l}(U)$ and thus $d_l = d^l$. To summarize, if $0 \leq l \leq s$ then

$$d_l = d_l^* = d^l.$$

In general, d_l and d_l^* can differ if $l > s$. For example, if $f[x_0 : \cdots : x_k] = [x_0^d : \cdots : x_k^d]$ and U is a small neighborhood of $A := \{x_{s+1} = \cdots = x_k = 0\}$ then for $s < l \leq k$

$$d_l = 0 \quad \text{and} \quad d_l^* = d^l.$$

A consequence of the convergence in Theorem 1.3.18 is that, if $\tau \in \mathscr{C}_p(U)$ is attracting on U then

$$\int_U (f^{n*}\omega^{s+1}) \wedge \omega^{p-1} = o(d^{sn})$$

and thus $d_{s+1} \leq d_s = d^s$ (see [Din07, Proposition 6.1]). Moreover, if the convergence in Theorem 1.3.18 has exponential speed, what I conjecture, then $d_{s+1} < d_s = d^s$. This would have consequences, in particular when $s = k - 1$ (see Theorem 1.4.4).

1.4.2 Small topological degree

When $l = k$ then it is easy to see that the inequality $d_k^* < d^s$ corresponds exactly to the following definition introduced by Daurat [Dau14, Definition 3.1].

Definition 1.4.1. *Let $U \subset \mathbb{P}^k$ be a trapping region of dimension s for f. The endomorphism f is said to be* asymptotically of small topological degree *on U if for all $p \in U$ we have* $\limsup_{n\to\infty}(\mathrm{card}(f^{-n}(p) \cap U))^{1/n} < d^s$. *In this situation, we say that the attracting set $A := \cap_{n\geq 0} f^n(U)$ is of* small topological degree.

Daurat proved that this notion is open in the set of endomorphisms and that if f is asymptotically of small topological degree on U then there exists $n_0 \geq 1$ such that for all $p \in U$, the number of preimages of p by f^{n_0} in U is smaller than $d^{n_0 s}$. In particular, an attracting set of small topological degree cannot be algebraic. Hence, the inequality $d_k^* < d^s$ gives a simple open condition which ensures that the corresponding attracting sets are not algebraic. For codimension 1 attracting sets, i.e. $s = k - 1$, she obtained a stronger result which was improved in [DT18a].

Theorem 1.4.2 ([Dau14, DT18a])**.** *Let $U \subset \mathbb{P}^k$ be a trapping region of dimension $s = k-1$ such that $d_k^* < d^s$. Then, every attracting current in $\mathscr{C}_1(U)$ has continuous local potentials. In particular, the attracting set associated to U cannot be pluripolar.*

Notice that the theorem in [DT18a] was stated under the assumptions of Dinh (HD). Nevertheless, using results of [Taf18], the proof can be extended to an arbitrary trapping region. In this proof, we study the potentials u_n of the push-forward of a smooth form $S \in \mathscr{C}_1(U)$ by f^n. An important point, which is a consequence of the codimension 1 assumption, is that these potentials are functions and not currents of higher bidegree. However, the key point about this codimension 1 assumption is much more basic. When we evaluate the function u_n on a positive measure m, this measure has an obvious decomposition

$$m = m_1 + m_2$$

where m_1 and m_2 are two positive measures (i.e. positive *closed* (k, k)-currents) supported on U and $\mathbb{P}^k \setminus U'$ respectively (where U' is a small neighborhood of $\overline{U}$). Without any assumption (see Proposition 1.5.2 in Section 1.5), we can obtain good estimates on $\langle u_n, m_2\rangle$ and the second part $\langle u_n, m_1\rangle$ is handled using the small topological degree assumption. In codimension $p \geq 2$, the potential of a smooth form $S \in \mathscr{C}_p(U)$ is a $(p-1, p-1)$-form which can be evaluated on currents $R \in \mathscr{C}_{k-p+1}(\mathbb{P}^k)$. However, a decomposition

$$R = R_1 + R_2$$

where R_1 and R_2 are two positive currents which are *closed* (or even dd^c-closed) supported respectively on U and $\mathbb{P}^k \setminus U'$ seems delicate to obtain when U is an arbitrary trapping region. We refer to Section 1.4.3 for a reformulation of the difficulties in higher codimensions and to Section 1.5.2 for a setting where the decomposition $R = R_1 + R_2$ with $dd^c R_1 = dd^c R_2 = 0$ holds.

A result similar to Theorem 1.4.2 has been obtained in [DS03] for polynomial-like maps. Actually, they proved that a similar inequality on degrees is equivalent to a regularity property of the dynamical object they considered (a measure in their situation). This equivalent can be extended to the setting of Theorem 1.4.2: if an attracting current $\tau \in \mathscr{C}_1(U)$ has continuous local potentials then there exists an attracting set $A' \subset U$ with $\tau \in \mathscr{C}_1(A')$ which is of small topological degree.

All this points out the interest of this framework. However, it is not clear how to obtain examples of attracting sets of small topological degree. One way to do this is to consider small perturbations of Hénon maps. Another one is given by a very nice construction of Daurat. In [Dau14], she exhibited algebraic conditions on a family of maps preserving a pencil of lines which ensure the existence of an attracting set of small topological degree. To be more precise, let $\mathscr{F}_d$ denote the set of pairs $f = (f_\infty, R)$ where R is a homogeneous polynomial of degree d in $\mathbb{C}^k$ and $f_\infty = (F_0, \dots, F_{k-1})$ is a k-tuple of homogeneous polynomials of degree d which defines a holomorphic endomorphism of $\mathbb{P}^{k-1}$.

Theorem 1.4.3 ([Dau14, DT18a])**.** *Let $k, d \geq 2$. There exists a non-empty Zariski open set $\Omega \subset \mathscr{F}_d$ such that if $(f_\infty, R) \in \Omega$ then for $\epsilon \in \mathbb{C}^*$ close enough to 0, the map f_ϵ given by*

$$f_\epsilon[x_0 : \cdots : x_{k-1} : x_k] = [f_\infty(x_0, \dots, x_{k-1}) : x_k^d + \epsilon R(x_0, \dots, x_{k-1})],$$

has a codimension 1 attracting set of small topological degree close to the hyperplane $\{x_k = 0\}$.

This theorem with $k = 2$ was obtained in [Dau14]. The construction in higher dimension is exactly the same and the only contribution of [DT18a] is to show that the Zariski open set Ω is also dense when $k \geq 3$ by giving examples.

A particularity of codimension 1 attracting sets of small topological degree is that the measures obtained in Corollary 1.3.21 are hyperbolic with 1 negative Lyapunov exponent et $k-1$ positive ones. This comes from an easy adaptation of the proof of [dT08] and it only requires the inequality $d_k < d^{k-1}$.

Theorem 1.4.4 ([DT18a])**.** *Let U be a codimension 1 trapping region. If $d_k < d^{k-1}$ then for every attracting current τ in $\mathscr{C}_1(U)$ then the measure $\nu_\tau := \tau \wedge T^{k-1}$ is hyperbolic with $k-1$ exponents larger than or equal to $1/2 \log d$ and one exponent smaller than or equal to $1/2 \log(d_k/d^{k-1})$.*

1.4.3 Higher codimensions

We conclude this section with a brief remark about attracting sets of higher codimension. The condition of being of small topological degree is well-adapted to attracting sets of codimension 1. In general, for a trapping region U of dimension s, the appropriated assumption on degrees in order to obtain regularity properties on attracting currents should be $d^*_{s+1} < d^s$. However, there are several difficulties to generalize the results above in this setting.

The first one is that when $s < k-1$ the local potentials of a current $\tau \in \mathscr{C}_p(U)$ ($s + p = k$) are $(p-1, p-1)$-currents which are not functions. Dinh and Sibony developed in [DS09] a very nice theory, called *super-potential theory*, in order to handle this kind of objects on $\mathbb{P}^k$. In particular, they extended the notion of positive closed $(1,1)$-currents with bounded (resp. continuous) local potentials to every bidegree with the notion of PB (resp. PC) positive closed (p,p)-currents. However, these definitions are global on $\mathbb{P}^k$ and are not necessarily well-adapted to our semi-local setting, i.e. in a trapping region U. An attempt of a more local definition is given in [Ahn18] but it doesn't correspond to our situation.

Nevertheless, one can show that the inequality $d^*_{s+1} < d^s$ on U implies that the attracting currents in $\mathscr{C}_p(U)$ satisfy a local version of PB (or PC). Conversely, if an attracting current τ satisfies (another) local version of PB (or PC) then $d^*_{s+1} < d^s$ on a trapping region containing $\mathrm{supp}(\tau)$. However, on an arbitrary trapping region, it is not clear if these two local versions of PB/PC are equivalent. One way to prove this is to solve the $\partial\overline{\partial}$-equation with estimates on U, which seems complicated to obtain in full generality. In the next section, we will see situations where such estimates hold.

Finally, notice that the lack of Khovanskii-Teissier-Gromov type inequality on the local dynamical degrees makes the generalization of Theorem 1.4.4 more difficult in higher codimension. A priori, we shall need the inequality $\max_{s+1\leq l\leq k} d_l < d^s$ to use directly the proof of [dT08].

1.5 Solutions of the $\partial\overline{\partial}$-equation and small Jacobians

Most of the known examples of attracting sets in $\mathbb{P}^2$ (see Section 1.6) are dissipative in the sense that $|\mathrm{Jac} f| < 1$ on the corresponding trapping regions. We will see in this section that this simple assumption gives a speed of convergence in Theorem 1.3.18 and in particular that the attracting current τ is the unique invariant current in its trapping region D_τ. This can be used to prove that the measure ν_τ obtained in Corollary 1.3.21 is the unique measure of maximal entropy in D_τ (see [Dau18]).

Exactly the same proof works for codimension 1 attracting sets in $\mathbb{P}^k$. For attracting sets of higher codimension, a similar speed of convergence can be obtained but it requires more involved techniques which necessitates additional assumptions on the geometry of the trapping region. Actually, we need to solve the $\partial\overline{\partial}$-equation with estimates on U, which can be done for example if U verifies a slightly stronger condition than (HD) (see (HD*)).

In this section, we start with the simpler case of codimension 1 attracting sets and we explain where our strategy requires a resolution of the $\partial\overline{\partial}$-equation. Then, we discuss how the version of Henkin-Leiterer of the theory of q-convex sets of Andreotti-Grauert helps to solve this equation (this has been done in [Taf13]).

1.5.1 Speed of convergence in codimension 1

In the following theorem, we require that the current τ is attracting for all the iterates of f. Recall that this is not a restrictive assumption by Proposition 1.3.35 and it is necessary to have the convergence $\Lambda^n R \to \tau$ in Theorem 1.3.18. Here, the operator Λ on $\mathscr{C}_p(\mathbb{P}^k)$ is defined by $\Lambda := d^{-s} f_*$ where $s := k - p$.

Theorem 1.5.1. *Let $\tau \in \mathscr{C}_1(\mathbb{P}^k)$ be a current attracting on a trapping region U which is also attracting for all the iterates f^n. If $|\mathrm{Jac} f| < d^{k-1}$ on U then there exist constances $c > 0$ and $0 < \lambda < 1$ such that*

$$|\langle \Lambda^n R - \tau, \phi\rangle| \leq c\lambda^n \|\phi\|_{\mathcal{C}^2}$$

for every $R \in \mathscr{C}_1(U)$ and every $\mathcal{C}^2$-form ϕ on $\mathbb{P}^k$. In particular, τ is the unique invariant current in $\mathscr{C}_1(U)$.

The idea of the proof is very simple. Let $l \geq 1$ be an arbitrary integer. If $R \in \mathscr{C}_1(U)$ and ϕ is a $\mathcal{C}^2$-form then we can write

$$\langle \Lambda^{(l+1)n} R - \tau, \phi \rangle = \left\langle \Lambda^{ln} R - \tau, \frac{1}{d^{(k-1)n}} f^{n*} \phi \right\rangle.$$

The assumption on the Jacobian of f implies that there exist constances $c_1 > 0$ and $0 < \lambda_1 < 1$ such that the $\mathcal{C}^0$-norm of $\frac{1}{d^{(k-1)n}} f^{n*}(dd^c\phi)$ is smaller than $c_1 \lambda_1^n \|\phi\|_{\mathcal{C}^2}$ on U. From this, there are two main steps.

Step 1: Resolution of the $\partial\overline{\partial}$-equation with estimates.

Recall that $dd^c = \frac{i}{\pi}\partial\overline{\partial}$. Hence, if we are able to solve the $\partial\overline{\partial}$-equation on U with good $\mathcal{C}^0$ estimates, there exist forms ψ_n on U such that

$$dd^c \psi_n = \frac{1}{d^{(k-1)n}} f^{n*}(dd^c\phi) \text{ on } U \quad \text{and} \quad \|\psi_n\|_{\mathcal{C}^0, U} \lesssim c_1 \lambda_1^n \|\phi\|_{\mathcal{C}^2}.$$

Step 2: Exponential speed of convergence for pluriharmonic observables.

As we have seen in Theorem 1.3.29, if $(R_\theta)_{\theta \in \mathbb{D}}$ is a structural disk in $\mathscr{C}_1(U)$ and Φ is a form on U with $dd^c\Phi = 0$ then the function

$$\theta \mapsto \langle R_\theta, \Phi \rangle$$

is harmonic on $\mathbb{D}$. Harmonic functions have much better compactness properties that subharmonic ones and each step in the proof of the convergence in Theorem 1.3.18 is easier when the observable is dd^c-closed. Actually, using Harnack's inequality it is possible to obtain an exponential speed for this convergence.

Proposition 1.5.2. *[Taf18, Proposition 3.37] Let τ and U be as in Theorem 1.5.1. Let $\mathcal{H}$ denote the set of continuous real $(k-1, k-1)$-forms Φ on U such that $dd^c\Phi = 0$ and $|\langle R - \tau, \Phi \rangle| \leq 1$ for all $R \in \mathscr{C}_1(U)$. There exist two constantes $c_2 > 0$ and $0 < \lambda_2 < 1$ such that for all $R \in \mathscr{C}_1(U)$, $\Phi \in \mathcal{H}$ and $n \geq 1$ we have*

$$|\langle \Lambda^n R - \tau, \Phi \rangle| \leq c_2 \lambda_2^n.$$

Notice that this result is indeed available for attracting sets of any dimension [Taf18, Proposition 3.37].

Step 3: End of the proof.

For $n \geq 1$, denote $\phi_n := \frac{1}{d^{(k-1)n}} f^{n*}(dd^c\phi)$ and let ψ_n be the form obtained in **Step 1**. Then for an integer $l \geq 1$ we have

$$\begin{aligned}\langle \Lambda^{(l+1)n} R - \tau, \phi \rangle &= \langle \Lambda^{ln} R - \tau, \phi_n \rangle \\ &= \langle \Lambda^{ln} R - \tau, \phi_n - \psi_n \rangle + \langle \Lambda^{ln} R - \tau, \psi_n \rangle \end{aligned} \tag{1.9}$$

An easy observation is that there exists a constant $M \geq 1$ such that for all continuous form Φ on $\mathbb{P}^k$ we have $\|d^{-(k-1)} f^* \Phi\|_\infty \leq M \|\Phi\|_\infty$ and thus $\|\phi_j\|_\infty \leq M^j \|\phi\|_\infty$ for all $j \geq 1$. Hence, since $dd^c(\phi_n - \psi_n) = 0$, Proposition 1.5.2 implies that

$$\langle \Lambda^{ln} R - \tau, \phi_n - \psi_n \rangle \lesssim \lambda_2^{ln} M^n \|\phi\|_\infty.$$

On the other hand, the estimate $\|\psi_n\|_{\mathcal{C}^0,U} \lesssim c_1\lambda_1^n\|\phi\|_{\mathcal{C}^2}$ implies that

$$\langle \Lambda^{ln}R - \tau, \psi_n\rangle \lesssim c_1\lambda_1^n\|\phi\|_{\mathcal{C}^2}.$$

Therefore, if $l \geq 1$ is large enough then $\lambda_2^l M < 1$ and both terms in (1.9) decrease exponentially fast with n, which implies the desired result. □

It remains to show how to complete **Step 1**. Actually, for codimension 1 attracting set, this is very easy. The key point is that $dd^c\phi_n$ is a (k,k)-form. Hence, if $0 \leq \chi \leq 1$ is a cut-off function supported in a small neighborhood of $\overline{U}$ and with $\chi = 1$ on U then $\chi dd^c\phi_n$ is still a closed form and we have good $\mathcal{C}^0$ estimates for it on the whole space $\mathbb{P}^k$. Hence, **Step 1** can be archived using a resolution of the dd^c-equation on $\mathbb{P}^k$ with estimates (see e.g. [DS09, Lemma 2.3.5] with $W' = W = \mathbb{P}^k$). In the general case, if U is a trapping region of dimension s with $s < k-1$ then the test form ϕ is a (s,s)-form and $dd^c\phi$ is a $(s+1,s+1)$-from. Therefore, there is no reason that $\chi dd^c\phi$ is still a dd^c-closed form. Hence, we need to solve the dd^c-equation with estimates directly on U, which is a delicate subject in general.

1.5.2 q-convex sets and Andreotti-Grauert theory

The goal of this subsection is to prove the following theorem. The strategy will be the same as the proof of Theorem 1.5.1. The aim of the assumption (*) is to allow us to complete **Step 1**. See below for explanations about this assumption.

Theorem 1.5.3. *Let τ be a current which is attracting on the codimension p trapping region U with respect to each iterate of f. Assume that the following conditions are satisfied:*

> *U is strictly $(p-1)$-convex and there exist two open sets U_1 and U_2 such that $U \subset U_1 \subset U_2$, U_1 is a deformation retract of a dimension s complex manifold $L \subset U_1$ and $\| \bigwedge^{s+1} Df(z)\| < d^s$ for all $z \in U_2$.* (*)

Then there exist constants $c > 0$ and $0 < \lambda < 1$ such that

$$|\langle \Lambda^n R - \tau, \phi\rangle| \leq c\lambda^n\|\phi\|_{\mathcal{C}^2}$$

for all $R \in \mathscr{C}_p(U)$ and all $\mathcal{C}^2$ test form ϕ. In particular, τ is the unique invariant current in $\mathscr{C}_p(U)$.

The three open sets U, U_1 and U_2 may coincide but it could be easier to check the three parts of (*) on three different sets. Although technical, the assumption (*) is satisfied in all the examples of Section 1.6 except for Theorem 1.6.8 and Theorem 1.6.9. Actually, a setting where the geometrical assumptions in (*) are verified (with $U = U_1$) is given by (HD*) below. It is a slight modification of (HD) and the examples of Section 1.6 (except Theorem 1.6.8) have trapping region with this property. The only difference with (HD) is that the slices of U have to be *strictly convex* and not just star-shaped. This was the framework of [Taf13].

> *There exist two linear subspaces I and L of dimension $p-1$ and s respectively such that $I \cap U = \varnothing$ and $L \subset U$. Moreover, for each $x \in L$ the unique dimension p linear subspace $I(x)$ containing I and x intersects U in a subset which is strictly convex in $I(x) \setminus I \simeq \mathbb{C}^p$.* (HD*)

Here are some additional remarks on (*).

- The condition $\|\bigwedge^{s+1} Df\| < d^s$ on U_2 is the exact counterpart to $|\mathrm{Jac} f| < d^{k-1}$ on U in Theorem 1.5.1.
- The deformation retraction of U_1 is used to solve the d-equation with estimates and to ensure that $H^{s,s+1}(U_1,\mathbb{C}) = 0$. It should be easy to weaken this condition.
- The strict $(p-1)$-convexity of U is the more technical part of (*). It is used to solve the $\overline{\partial}$-equation with estimates on U using the results of Henkin-Leiterer [HL88]. However, it turns out that this condition might be always satisfied in a larger trapping region (see **(2)** in Remark 1.5.6).

We refer to [HL88] and [Dem12] for the precise definition of q-convex sets and for further developments. We will use the conventions of [HL88] but observe that q-convex domains in [HL88] correspond to strongly $(k-q)$-convex ones in [Dem12]. Loosely speaking, a function u on a complex manifold M is *q-convex* if at every point there exists a q-dimensional complex submanifold Y such that $u_{|Y}$ is strictly plurisubharmonic. And the definition of strictly q-convex domains mimics the one of strictly pseudoconvex domains by replacing strictly plurisubharmonic functions by $(q+1)$-convex functions: a relatively compact open set D of M is *strictly q-convex* if there exists a neighborhood Ω of ∂D and a $(q+1)$-convex function u on Ω such that

$$D \cap \Omega = \{x \in \Omega \,|\, u(x) < 0\}.$$

When Ω can be chosen to be a neighborhood of $\overline{D}$ then D is said to be *completely strictly q-convex.* For information, notice that Andreotti and Grauert obtained the following vanishing theorem which generalized in some sense Theorem B of Cartan.

Theorem 1.5.4 ([AG62])**.** *If D is a completely strictly q-convex open subset of $\mathbb{P}^k$ with $\mathcal{C}^2$ boundary then $H^{s,r}(D,\mathbb{C}) = 0$ for any $0 \le s \le k$ and $k-q \le r \le k$.*

Henkin-Leiterer revisited the results of [AG62] in [HL88] using integral representations. One of the main avantages of [HL88] is that it gives explicit estimates for the $\overline{\partial}$-equation.

Theorem 1.5.5. *[HL88, Theorem 11.2] Let D be a strictly q-convex open subset of $\mathbb{P}^k$ with $\mathcal{C}^2$ boundary. If ϕ is a continuous $\overline{\partial}$-exact form of bidegree (r,s) in a neighborhood of $\overline{D}$ with $0 \le s \le k$, $k-q \le r \le k$, then there exists a continuous $(s, r-1)$-form ψ on D such that $\overline{\partial}\psi = \phi$ and*

$$\|\psi\|_{\infty,D} \le C\|\phi\|_{\infty,D}$$

for some $C > 0$ independent of ϕ.

Using the method of Dinh-Nguyen-Sibony [DNS08, Theorem 2.7], Theorem 1.5.5 and the others conditions in (*) give a resolution of the $\partial\overline{\partial}$-equation with estimates which archives **Step 1**. The rest of the proof of Theorem 1.5.3 is identical to the one of Theorem 1.5.1.

Remark 1.5.6. ***(1)*** *Let $0 \le p, s \le k$ be such that $s + p = k$. To an open set $U \subset \mathbb{P}^k$ it is possible to associate two open sets $\widetilde{U}$ and $\widehat{U}$ that we call the s-pseudoconcave core of U and the $(p-1)$-pseudoconvex hull of U respectively. The former is defined by*

$$\widetilde{U} := \bigcup_{S \in \mathscr{C}_p(U)} \mathrm{supp}(S).$$

The latter $\widehat{U}$ is the complement of

$$\overline{\cup_{S \in \mathscr{C}_{s+1}(\mathbb{P}^k \setminus U)} \mathrm{supp}(S)}.$$

They satisfy $\widetilde{U} \subset U \subset \widehat{U}$. *Both these sets have interesting geometric properties. When* $s = 0$ *then* $\widehat{U}$ *is the rationally convex hull of* U, *see [Gue99] and [DS95]. It is easy to check that if* U *is a trapping region of dimension* s *then this is also the case for both* $\widetilde{U}$ *and* $\widehat{U}$. *In some sense, the attracting set associated to* $\widetilde{U}$ *can be seen as the part of* pure dimension s *of the one associated to* U.

(2) It is likely that slight modifications of $\widehat{U}$ *give codimension* p *trapping region which are strictly* $(p-1)$*-convex, i.e. the assumption on* U *in Theorem 1.5.3 should always be satisfied on a lager trapping region. The idea is the following. It is easy to construct a current* $S \in \mathscr{C}_{s+1}(\mathbb{P}^k)$ *such that* $\widehat{U} = \mathbb{P}^k \setminus \mathrm{supp}(S)$. *By [FS95b], this gives to* $\widehat{U}$ *a convexity property called* $(p-1)$*-pseudoconvexity. If* S *can be slightly modified in a current* S' *such that* $\widehat{U}' := \mathbb{P}^k \setminus \mathrm{supp}(S')$ *has a smooth boundary (which is not totally clear), then* $\widehat{U}'$ *is a trapping region and the regularity of the boundary turns the* $(p-1)$*-pseudoconvexity of* $\widehat{U}'$ *into* $(p-1)$*-convexity (see [Mat93]). Taking a slightly smaller open set, we obtain a strictly* $(p-1)$*-convex trapping region which contains* U.

1.6 Examples

This section gathers most of the known examples of attracting sets and attractors. Notice that the inclusion $\overline{f(U)} \subset U$ is stable by small perturbations of f, thus perturbations of these examples also admits attracting sets. Moreover, by a result of Fakhruddin [Fak14], these attracting sets are not algebraic for a very generic perturbation. However, with regards to attractors, the only known examples which are not algebraic are highly non-generic since they all preserve a fibration. Nevertheless, it is very likely that there exist open sets of endomorphisms which admit attractors that are not attracting cycles.

The most basic example of an attracting set is the lift of an endomorphism of $\mathbb{P}^s$ to one of $\mathbb{P}^k$, $s < k$.

Example 1.6.1. *Let* g *be an endomorphism of* $\mathbb{P}^s$ *of degree* $d \geq 2$ *given in homogeneous coordinates by*

$$g[x_0 : \cdots : x_s] = [G_0(x_0, \ldots, x_s) : \cdots : G_s(x_0, \ldots, x_s)].$$

Then, the endomorphism of $\mathbb{P}^k$ *defined by*

$$f[x_0 : \cdots : x_k] = [G_0(x_0, \ldots, x_s) : \cdots : G_s(x_0, \ldots, x_s) : x_{s+1}^d : \cdots : x_k^d]$$

admits the linear subspace $A = \{x_{s+1} = \cdots = x_k = 0\}$ *as an attracting set of dimension* s. *The restriction of* f *to* A *corresponds to* g *and can be chosen to be topologically transitive. In this case,* A *is an attractor.*

A similar idea can give several disjoint attractors.

Example 1.6.2. *Let* g *be a rational mapping of* $\mathbb{P}^1$ *whose Julia set is the whole space* $\mathbb{P}^1$. *There exist two homogeneous polynomials* $P, Q \in \mathbb{C}[x, y]$ *such that, in homogeneous coordinates on* $\mathbb{P}^1$

$$g[x : y] = [P(x, y) : Q(x, y)].$$

The self-map f *of* $\mathbb{P}^3$ *given by*

$$f[x : y : z : t] = [P(x, y) : Q(x, y) : P(z, t) : Q(z, t)]$$

is holomorphic and admits the open sets

$$U_1 := \{[x : y : z : t] \in \mathbb{P}^3 \mid \max(|x|, |y|) < \epsilon \max(|z|, |t|)\}$$

and

$$U_2 := \{[x : y : z : t] \in \mathbb{P}^3 \mid \max(|z|, |t|) < \epsilon \max(|x|, |y|)\}$$

as disjoint trapping regions when $\epsilon > 0$ is small enough. The associated attracting sets are respectively $A_1 = \{x = y = 0\}$ and $A_2 = \{z = t = 0\}$. The restriction of f to one of these sets is given by g and thus is topologically transitive, i.e. A_1 and A_2 are disjoint attractors of $\mathbb{P}^3$.

On can perturbe maps given in Example 1.6.1 with $s = k - 1$ in order to have non-algebraic attractors (see [JW00] and [FS01] for $k = 2$ and [Ron12] for $k \geq 3$).

Theorem 1.6.3. *For $\lambda \in \mathbb{C}$, let f_λ be the endomorphism of $\mathbb{P}^k$ defined by*

$$f_\lambda[z : w_1 : \cdots : w_{k-1} : t] = [(z - 2w_1)^2 : \cdots : (z - 2w_{k-1})^2 : z^2 : t^2 + \lambda z^2].$$

If $|\lambda| \neq 0$ is small enough then f_λ has a non-algebraic attractor K_λ close to the hyperplane $\{t = 0\}$.

The key point is the proof of the transitivity of $(f_\lambda)_{|K_\lambda}$ is that f_λ preserves the pencil of lines passing through $[0 : \cdots : 0 : 1]$ and acts on it in a transitive way. All the known examples of non-algebraic attractors use this idea.

Although [Fak14] implies that for a very generic endomorphism of $\mathbb{P}^k$ all attracting sets of dimension s, with $0 < s < k$, are Zariski dense, it is not easy to exhibit an explicit example in codimension larger than 1. This has been done by Daurat.

Theorem 1.6.4. *[Dau14, Theorem 5.7] If $\epsilon_1, \epsilon_2 \in \mathbb{C}^*$ are such that $|\epsilon_2| << |\epsilon_1| << 1$ then the endomorphism of $\mathbb{P}^3$ defined by*

$$f[x : y : z : t] = [x^2 + 0.1y^2 : y^2 : z^2 + \epsilon_1(x^2 + xy) : t^2 + \epsilon_2(x^2 + xy)]$$

has a Zariski dense attracting set of dimension 1 which is close to the line $\{z = t = 0\}$.

Another general way to obtain attracting sets is the following construction, inspired by [BD02b, Theorem 4.1].

Example 1.6.5. *Let $g\colon \mathbb{P}^s \to \mathbb{P}^k$ be a holomorphic map and define $A_0 := g(\mathbb{P}^s)$. Let $I \subset \mathbb{P}^k$ be a linear subspace of dimension $p-1$ such that $I \cap A_0 = \varnothing$. If $L \subset \mathbb{P}^k$ is a linear subspace of dimension s then we can consider the projection $\pi : \mathbb{P}^k \setminus I \to L$. By identifying L with $\mathbb{P}^s$ we obtain a map $f_0 := g \circ \pi\colon \mathbb{P}^k \setminus I \to \mathbb{P}^k$ with $f_0(A_0) = A_0$ and which has rank s. In particular, $\bigwedge^{s+1} Df_0 = 0$ everywhere. A generic small perturbation of f_0 gives a holomorphic map $f\colon \mathbb{P}^k \to \mathbb{P}^k$ which has an attracting set A near A_0.*

Observe that for these examples $\bigwedge^{s+1} Df$ is very small on A. It implies easily that the measure ν obtained by Corollary 1.3.21 is hyperbolic with s positive exponents and $p := k - s$ negative ones. Another observation is that if the perturbation of f_0 is small enough then A admits a trapping region which satisfies (HD*). Indeed, if Ω is a small neighborhood of I then $U := \mathbb{P}^k \setminus \overline{\Omega}$ is a trapping region for f_0 and thus for small perturbations of f_0. And, Ω can easily be chosen such that U satisfies (HD*).

Here is an implementation of the construction given in Example 1.6.5 (see [FW99] and [Dau18]).

Example 1.6.6. *Let X be the smooth conic in $\mathbb{P}^2$ given by $X = \{xy = z^2\}$. It is the image of $g[a : b] = [a^2 : b^2 : ab]$ and if $I = [0 : 0 : 1]$ and $L = \{z = 0\}$ then the map f_0 in Example 1.6.5 is*

$$f_0[x : y : z] = [x^2 : y^2 : xy].$$

Perturbations of the form

$$f_\epsilon[x : y : z] = [x^2 : y^2 : xy + \epsilon(xy - z^2)]$$

with $|\epsilon| \neq 0$ *small give endomorphisms of* $\mathbb{P}^2$ *admitting* X *as an attracting set. The induced map on* $X \simeq \mathbb{P}^1$ *is* $[x^2 : y^2]$. *If we take* $g[a : b] = [(a - 2b)^2 : a^2 : a(a - 2b)]$ *then*

$$f_\epsilon[x : y : z] = [(x - 2y)^2 : x^2 : x(x - 2y) + \epsilon(xy - z^2)]$$

admits X *as an attractor for* $|\epsilon| \neq 0$ *small enough.*

A last simple example is given by perturbations of invariant critical hypersurfaces. They are the counterparts in higher dimension of super-attractive fixed points in dimension 1.

Example 1.6.7. *Let* f_0 *be in an endomorphism of* $\mathbb{P}^k$. *Assume there exists a hypersurface* H *which is* f_0*-invariant and included in the critical set of* f_0. *If* $k = 2$ *or if* H *is smooth then* H *is an attracting set for* f_0 *by [FS94a, Lemma 7.9] and [Sta06]. In both cases, Theorem 1.5.1 applies to small perturbations of* f_0.

The following theorem gives an example of a quasi-attractor K in $\mathbb{P}^2$ which is not an attracting set, i.e. $K = \cap_{n\geq 0} A_n$ where $(A_n)_{n\geq 0}$ is a strictly decreasing sequence of attracting sets (see [Taf17]). Notice that results on attracting currents from [Taf18] imply that the Hausdorff dimension of A_n must be larger than or equal to 3. In this example, each A_n contains repelling cycles and thus has non-empty interior and Hausdorff dimension equal to 4. To the best of my knowledge, it is the only known examples of such quasi-attractors and it turns out that exactly the same maps were consider by Fornæss-Sibony (see [FS01, Theorem 4.3]).

Theorem 1.6.8. *If* f_0 *is a volume increasing polynomial automorphism of* $\mathbb{C}^2$ *with a robust tangency then a generic (in the sense of Baire category) small perturbation* f *of* f_0 *is an endomorphism of* $\mathbb{P}^2$ *possessing uncountably many quasi-attractors which are not attracting sets. Such a perturbation* f *also has an attracting set* $A \neq \mathbb{P}^2$ *with infinitely many repelling periodic points.*

A map f_0 satisfying these assumptions is given in [Buz97]. The idea of the proof of Theorem 1.6.8 is very simple. The ingredients are the following.

- f_0 is injective on $\mathbb{C}^2$,
- all its tangencies lie in a large polydisk $\mathbb{B} \subset \mathbb{C}^2$,
- f_0 has an attracting set of dimension 1, usually denoted by K^-, which possesses a trapping region U containing $\mathbb{B}$.

Hence, a small perturbation f of f_0 still admits U as a trapping region and is injective on $\mathbb{B}$. For a generic f, the construction of Newhouse gives infinitely many repelling cycles in $\mathbb{B}$. The injectivity of f implies that if Ω is a small neighborhood of a finite union of such cycles then $U \setminus \overline{\Omega}$ is a trapping region. Taking more and more cycles, we obtain a strictly decreasing and infinite sequence of attracting sets. Observe that the attracting set associated to $U \setminus \overline{\Omega}$ cannot have a trapping region satisfying (HD).

Theorem 1.6.8 gives examples of attracting sets with non-empty interior but they cannot be minimal as they possess an attracting fixed point near the line at infinity. Using a totally different method, called blenders, we obtain in [Taf17] an example of an attractor of $\mathbb{P}^2$, distinct from $\mathbb{P}^2$, with non-empty interior.

Theorem 1.6.9. *There exists an endomorphism f of $\mathbb{P}^2$ which has an attractor A which contains an algebraic curve in its interior.*

In particular, such an attractor is not rigid in the sense of Dinh-Sibony (see [DS14]), i.e. the set $\mathscr{C}_1(A)$ contains infinitely many different currents. The blenders used in this construction are called of *saddle type* in [Taf17] and are indeed very simple objects. Roughly speaking, a saddle blender corresponds to an open set Z such that

(1) $\overline{Z} \subset f(Z)$,

(2) $\Lambda := \cap_{n\geq 0} f^{-n}(\overline{Z})$ is a saddle hyperbolic set.

Their saddle nature allows us to easily find examples of trapping regions U containing such open set Z. Then the point **(1)** insures that Z is in the interior of $A := \cap_{n\geq 0} f^n(U)$. The rest of the proof of Theorem 1.6.9 is technical but elementary. As a last observation, note that the point **(1)** is stable by small perturbations so all the endomorphisms in a neighborhood of f have a proper attracting set with non-empty interior.

Chapter 2

Bifurcations in complex dynamics

The study of bifurcation phenomena for holomorphic families of rational mappings of $\mathbb{P}^1$ is a well-established and rich subject. It started with the seminal papers of Mañé-Sad-Sullivan [MSS83] and Lyubich [Lyu83b] (see Section 2.1). A nice counterpart in higher dimension to this theory has been recently obtained by Berteloot-Bianchi-Dupont [BBD18] (see Section 2.2). My main contributions in this topic have been to exhibit phenomena appearing only in dimension larger than or equal to 2. In [BT17], I studied with Bianchi a very specific family of endomorphisms of $\mathbb{P}^2$ with two interesting properties regarding bifurcations (see Section 2.3):

- the Julia set moves continuously on open sets of parameters containing bifurcations,
- the bifurcation locus of this family has non-empty interior.

This answers two questions from [BBD18]. At about the same time, Dujardin gave in [Duj17] two general mechanisms which lead to open sets of bifurcations in the spaces $\mathscr{H}_d(\mathbb{P}^k)$ of all degree d endomorphisms of $\mathbb{P}^k$, $k \geq 2$. The first one is based on a topological argument while the second relies on a construction coming from smooth dynamics called *blender*, originally introduced by Bonatti-Díaz in [BD96]. It turns out that, with Bonatti, we were also working on this question using blenders. This leads to a quick answer in [Taf17] to a conjecture of Dujardin [Duj17] (see Section 2.4).

Another direction, which will not be mentioned here and which still need to be developed, is the bifurcations of attracting sets. Some very partial results can be deduced from the results of Section 1.3 (see [Taf18, Section 5.2]). In particular, the sum of the Lyapunov exponents of the equilibrium measure supported on an attracting set depends in a plurisubharmonic way on the parameters. It is then very natural to wonder which dynamical interpretation can have the current obtained as the dd^c of this function.

2.1 Stability and bifurcations on $\mathbb{P}^1$

In general, there exists many different ways to detect the lacks of structural stability. Among them, there are

- a change in the type (attracting, repelling, saddle etc.) of a periodic point,
- a change in the structure of attracting sets,
- a change in other kinds of invariant sets (non-wandering set, supports of measures of maximal entropy etc.),

- a change in the critical dynamics (if any).

A remarkable feature of one variable complex dynamics is that (almost) all these types of bifurcations coincide for all holomorphic families of rational mappings of $\mathbb{P}^1$. More precisely, the study of bifurcations in such families is based on the following result.

Theorem 2.1.1 ([MSS83], [Lyu83b]). *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of rational mappings of degree $d \geq 2$ of $\mathbb{P}^1$. Let Ω be a connected open subset of M. The following are equivalent:*

1) *for all $\lambda \in \Omega$, the periodic points of f_λ can be followed holomorphically in Ω and do not change type,*

2) *the number and/or the period of attracting cycles are constant in Ω,*

3) *the map $\lambda \mapsto J_{f_\lambda}$ is continuous on Ω with respect to the Hausdorff topology on compact subsets of $\mathbb{P}^1$,*

4) *there is no* Misiurewicz parameter *in Ω,*

5) *for all $\lambda, \lambda' \in \Omega$, $(f_\lambda)_{|J_{f_\lambda}}$ is topologically conjugated to $(f_{\lambda'})_{|J_{f_{\lambda'}}}$.*

Definition 2.1.2. *Let $(f_\lambda)_{\lambda\in M}$ be as in Theorem 2.1.1. The union of all the open subsets of M where the points* ***1)-5)*** *of this theorem hold is called the* stability locus *of the family $(f_\lambda)_{\lambda\in M}$ and is denoted by* Stab. *Its complementary* Bif $:= M \setminus$ Stab *is the* bifurcation locus *of the family.*

Here, a parameter $\lambda \in \Omega$ is called *Misiurewicz* if f_λ possesses a repelling point r which is in the orbit of a critical point in a non-persistent way, i.e. there exists $\lambda' \in \Omega$ close to λ such that the holomorphic continuation of r for $f_{\lambda'}$ is not in the postcritical set of $f_{\lambda'}$. This definition corresponds to the one given in [BBD18] for $\mathbb{P}^k$ (see Definition 2.2.2). It doesn't coincide with the homonymous notion usually used in one variable complex dynamics.

Remark 2.1.3. *Misiurewicz parameters correspond to one type of bifurcations in the dynamics of critical points. There exists other types of bifurcations in the critical dynamics which can belong to the stability locus. For instance, in the family $(f_\lambda)_{\lambda\in\mathbb{C}}$ defined by $f_\lambda(z) = z^2 + \lambda$, the unique critical point is fixed for $\lambda = 0$ and has an infinite orbit for $\lambda \neq 0$ close to* 0.

Remark 2.1.4. *The points* ***3)*** *and* ***5)*** *in the theorem are consequences of the fact that the Julia set moves under a holomorphic motion in Ω. In one complex variable it is easy to extend a holomorphic motion of a set to its closure and thus this holomorphic motion of the Julia set can be obtained as the extension of the motion of the repelling points. In higher dimension, this extension result is no longer available and, as we will see in the next section, the counterpart to* ***5)*** *is not known.*

The following important result can be deduced form Theorem 2.1.1 and the fact that f_λ has at most $2d-2$ critical points.

Theorem 2.1.5 ([MSS83], [Lyu83b]). *For every holomorphic family $(f_\lambda)_{\lambda\in M}$ of holomorphic mappings of $\mathbb{P}^1$, the stability locus is dense in M.*

As we already mentioned, this result has been improved by McMullen-Sullivan [MS98]. They showed that the only obstructions to extend the conjugation given in Theorem 2.1.1 **5)** to the whole space $\mathbb{P}^1$ are bifurcations in the critical dynamics.

Theorem 2.1.6 ([MS98]). *Let $(f_\lambda)_{\lambda\in M}$ be as in Theorem 2.1.1. There exists a dense open subset X^{top} of M such that for every λ, λ' in the same connected component of X^{top} f_λ is topologically conjugated to $f_{\lambda'}$.*

If f is a rational mapping of $\mathbb{P}^1$ of degree $d \geq 2$ then Lyubich [Lyu83a] and Mañé [Mañ83] have independently proved that the equilibrium measure of f is the unique measure of maximal entropy $\log d$. Since the entropy of a measure is invariant by conjugation, any conjugation between $(f_\lambda)_{|J_{f_\lambda}}$ and $(f_{\lambda'})_{|J_{f_{\lambda'}}}$ has to map the equilibrium measure of f_λ to the one of $f_{\lambda'}$, i.e. the equilibrium measure is a natural object to consider in the study of bifurcations. Hence, one can wonder whether there exists an ergodic counterpart to the points **1)-5)** in Theorem 2.1.1. It turns out that DeMarco obtained such a characterization of the stability locus using the Lyapunov exponent of the equilibrium measure.

Theorem 2.1.7 (DeMarco [DeM03]). *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of rational mappings of degree $d \geq 2$ of $\mathbb{P}^1$. Let $L(\lambda)$ denote the Lyapunov exponent of the equilibrium measure μ_λ of f_λ. An open subset $\Omega \subset M$ is contained in the stability locus if and only if $\lambda \mapsto L(\lambda)$ is pluriharmonic on Ω.*

Actually, using the equidistribution of the repelling periodic points of f_λ towards μ_λ it is easy to see that $L(\lambda)$ is a plurisubharmonic function on M. Hence, the $(1,1)$-current $T_{\mathrm{Bif}} := dd^cL$ is positive and an equivalent formulation of Theorem 2.1.7 is that $\mathrm{supp}(T_{\mathrm{Bif}})$ coincides exactly with the bifurcation locus of the family $(f_\lambda)_{\lambda\in M}$. Moreover, this result allows to study bifurcations using pluripotential methods which look like those used in higher dimension complex dynamics. This gives rise to a very active and rich field of research (see e.g. [BB07], [BB09], [DF08], [GOV19]). As we will see in the next section, Theorem 2.1.7 also gives a indication on a good way to generalize Theorem 2.1.1 in higher dimension.

2.2 Theory in higher dimension

On $\mathbb{P}^k$, $k \geq 2$, the dynamics of a single endomorphism is not well understood thus it seems unrealistic to attempt a global bifurcation theory. Nevertheless, the properties of the equilibrium measure are essentially the same in all dimensions. Hence, its support, the *small Julia set* $\mathscr{J}_k$, could be a good candidate to replace the Julia set in dimension one.

For the rest of the section, $(f_\lambda)_{\lambda\in M}$ is a holomorphic family of endomorphisms of $\mathbb{P}^k$ of degree $d \geq 2$. The equilibrium measure of f_λ is denoted by μ_λ and we set $\mathscr{J}_k(\lambda) := \mathrm{supp}(\mu_\lambda)$. Briend-Duval had proved [BD99] that the Lyapunov exponents $\chi_1(\lambda), \ldots, \chi_k(\lambda)$ of μ_λ are all positive and bounded from below by $2^{-1}\log d$. By a result of Dinh-Sibony [DS03] on polynomial-like maps the sum of these exponents

$$L(\lambda) := \sum_{i=1}^{k} \chi_i(\lambda)$$

defined a plurisubharmonic function on M and thus it is natural to wonder what is the dynamical interpretation of the positive closed $(1,1)$-current dd^cL on M. This study started with the work [BB07] of Bassanelli-Berteloot where they proved that $dd^cL = 0$ when the repelling cycles move holomorphically. They also obtained formulae expressing L and dd^cL in terms of currents in $M \times \mathbb{P}^k$. Actually, Pham proved independently in [Pha05] that if $\mathcal{E}$ is a positive closed (k,k)-current in $M \times \mathbb{P}^k$ whose slices over each $\lambda \in M$ is μ_λ then

$$dd^cL := \pi_*(\mathcal{E} \wedge [C_f]), \tag{2.1}$$

where C_f is the critical set of the map $(\lambda, z) \mapsto (\lambda, f_\lambda(z))$ and π is to canonical projection from $M \times \mathbb{P}^k$ to M. The formula of Bassanelli-Berteloot corresponds to (2.1) with a particular choice of $\mathcal{E}$.

Amount other things, (2.1) clearly links the current $dd^c L$ with interplays between the critical set and the equilibrium measure. This formula is the key ingredient in the following theorem of Berteloot-Bianchi-Dupont which can be considered as the founding result in the theory of bifurcations in $\mathbb{P}^k$.

Theorem 2.2.1 ([BBD18])**.** *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of endomorphisms of $\mathbb{P}^k$ of degree $d \geq 2$ parametrized by a simply connected complex manifold M. Assume moreover that*

(*) $(f_\lambda)_{\lambda\in M}$ forms an open subset of the space $\mathscr{H}_d(\mathbb{P}^k)$ of all degree d endomorphisms of $\mathbb{P}^k$,

or

(**) $k = 2$.

Then, the following are equivalent:

1) *for all $\lambda \in M$, the repelling periodic points of f_λ in $\mathscr{J}_k(\lambda)$ can be followed holomorphically in M and do not change type,*

2) *the function L is pluriharmonic on M (i.e. $dd^c L = 0$ on M),*

3) *there is no* Misiurewicz parameter *in M,*

4) *there exists an* equilibrium lamination *over M.*

The precise definition of a Misiurewicz parameter in this context is the following.

Definition 2.2.2. *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of endomorphisms of $\mathbb{P}^k$ of degree $d \geq 2$. We denote by f the self-map of $M \times \mathbb{P}^k$ defined by $f(\lambda, z) := (\lambda, f_\lambda(z))$ and by C_f its critical set. A parameter λ_0 in M is called* Misiurewicz *if there exist an integer $n_0 \geq 0$, a neighborhood $N \subset M$ of λ_0 and a holomorphic map $\gamma\colon N \to \mathbb{P}^k$ such that:*

1. *for all $\lambda \in N$, $\gamma(\lambda)$ is a repelling periodic point of f_λ belonging to $\mathscr{J}_k(\lambda)$,*
2. *$(\lambda_0, \gamma(\lambda_0))$ belongs to an irreducible component X of $f^{n_0}(C_f)$,*
3. *the graph of γ is not contained in X.*

The bifurcation locus in the sense of Theorem 2.2.1 of the family $(f_\lambda)_{\lambda\in M}$ can then be defined as the closure of the Misiurewicz parameters.

We will not give an explicit definition of an equilibrium lamination but in particular **4)** implies that for all $\lambda, \lambda' \in M$ there exists a continuous conjugation between f_λ and $f_{\lambda'}$ on subsets of full measure for μ_λ and $\mu_{\lambda'}$ respectively (see [BD17] for a similar result for polynomial automorphisms of $\mathbb{C}^2$). Here are some additional remarks on this theorem.

- The assumptions (*) and (**) are technical and could probably be removed. Some progresses in that direction have been obtained by Bianchi [Bia19].
- Although the equilibrium lamination gives a conjugation between full measure subsets with respect to the equilibrium measures, it is not known whether or not $(f_\lambda)_{|\mathscr{J}_k(\lambda)}$ is topologically conjugated to $(f_{\lambda'})_{|\mathscr{J}_k(\lambda')}$ $(\lambda, \lambda' \in M)$, even in (a priori) simpler cases like polynomial skew products of $\mathbb{C}^2$.

- Actually, because of the non-injectivity and of the large critical set, it is not easy to obtain conjugacies in our context. For example, as shown by Jonsson [Jon98], a saddle hyperbolic set might not be structurally stable. And it is an open problem, probably feasible, to exhibit a structurally stable endomorphism in the family $\mathscr{H}_d(\mathbb{P}^k)$ of all degree d endomorphisms of $\mathbb{P}^k$, when $k \geq 2$.
- Beyond conjugacies, it is natural to wonder whether other one-dimensional results given in the previous section can still be valid in this context. In particular, the continuity of $\lambda \mapsto \mathscr{J}_k(\lambda)$ in a stable family is obtained in [BBD18] but the converse is left open. Similarly, it was not known at that time whether the stability locus is dense in all families. The purpose of [BT17] (see Section 2.3) is to give an explicit and simple counter-example to both these questions.
- Another natural question is whether stability in the sense of Theorem 2.2.1 is compatible with other kinds of bifurcations like Newhouse phenomenon. Dujardin answers this question in [Duj17] by giving a stable open subset of $\mathscr{H}_d(\mathbb{P}^2)$ (with d very large) with such phenomenon. In particular, bifurcations of repelling periodic points outside $\mathscr{J}_k$ can happen in a stable region in the sense of Theorem 2.2.1.

2.3 The elementary Desboves family

As we already said, the purpose of this section is to give a counter-example to the two following questions:

- If the small Julia set $\mathscr{J}_k$ depends continuously of the parameter in a family, is this family stable (i.e. satisfies **1)-4)** in Theorem 2.2.1)?
- Is the stability locus dense in every holomorphic family of endomorphisms of $\mathbb{P}^k$?

Actually, the counter-example given in [BT17] is a family already studied in [BD02b] and [BDM07] and these authors referred to it as *the elementary Desboves family.* They considered these maps because each of them preserves the Fermat curve

$$\mathcal{F} := \{x^3 + y^3 + z^3 = 0\}$$

and some of them possess statistical attractors with *intermingled basins* (see [BDM07, Section 6]).

Theorem 2.3.1 ([BT17])**.** *The family of endomorphisms of* $\mathbb{P}^2$ *defined by*

$$f_\lambda[x : y : z] = [-x(x^3 + 2z^3) : y(z^3 - x^3 + \lambda(x^3 + y^3 + z^3)) : z(2x^3 + z^3)]$$

with $\lambda \in \mathbb{C}^*$ *satisfies the following properties:*

- *the small Julia set of* f_λ *depends continuously on* λ*, for the Hausdorff topology,*
- *the bifurcation locus coincides with* $\mathbb{C}^*$ *(i.e. the stability locus is empty).*

The proof of this result is simple and we now give the main arguments. The first step is the following result which collects information on the maps in the elementary Desboves family obtained in [BD02b].

Theorem 2.3.2 ([BD02b])**.** *Let* $\lambda \in \mathbb{C}^*$*. The map* f_λ *in the elementary Desboves family satisfies the following properties:*

1. *f_λ preserves the line $Y := \{y = 0\}$ and $(f_\lambda)_{|Y}$ is a Lattès map,*
2. *f_λ preserves the pencil of lines passing through the point $\rho_0 := [0 : 1 : 0]$ and acts on it as $(f_\lambda)_{|Y}$ (in particular, transitively),*
3. *ρ_0 is an attracting fixed point and the Fatou set of f_λ is reduced to the basin of ρ_0,*
4. *the Julia set and the small Julia set of f_λ coincide, i.e. $\mathscr{J}_1(f_\lambda) = \mathscr{J}_2(f_\lambda)$.*

Proof. The two first points are simple verifications. The third one is a consequence of the second, i.e. f_λ acts in a chaotic way on a pencil of lines so an open subset whose images by f_λ^n stay away from the center of the pencil cannot belong to the Fatou set.

The last point is based on the fact that in such fibered situation in $\mathbb{P}^2$, the small Julia set is equal to the closure of the set of repelling points. One inclusion ($\mathscr{J}_2(f_\lambda)$ is contained in this closure) is a direct consequence of the equidistribution of repelling periodic points towards the equilibrium measure obtained in [BD99]. The other inclusion is not true in general but it has been obtained by Jonsson for polynomial skew products of $\mathbb{C}^2$ in [Jon99] and for the other endomorphisms preserving a pencil of lines in $\mathbb{P}^2$ in [Dab00]. A simple proof of this fact can be deduced from [DT18b] (see Corollary 3.1.3 in Chapter 3). From this, the rest of the proof goes as follow. If r is one of the three intersection points between Y and the Fermat curve $\mathcal{F} := \{x^3 + y^3 + z^3 = 0\}$ (for example, $r = [1 : 0 : -1]$) then it is easy to check that r is a repelling fixed point. In particular, $r \in \mathscr{J}_2(f_\lambda)$. Since $(f_\lambda)_{|Y}$ is a Lattès map, the preimages of r are dense in Y and thus the totally invariance of $\mathscr{J}_2(f_\lambda)$ and its closeness imply that $Y \subset \mathscr{J}_2(f_\lambda)$. Finally, Y is not totally invariant so by the equidistribution of hypersurfaces towards the Green current [DS08],

$$\mathscr{J}_1(f_\lambda) \subset \overline{\cup_{n\geq 1} f_\lambda^{-n}(Y)} \subset \mathscr{J}_2(f_\lambda) \subset \mathscr{J}_1(f_\lambda),$$

which gives the desired result. □

Proof of Theorem 2.3.1. A classical fact about the small Julia set is that it depends lower semi-continuously of the parameter. This is due to the fact that $\lambda \mapsto \mu_\lambda$ is continuous for the weak topology of measures and that the support function is lower semi-continuous. On the other hand, it is a general fact in continuous dynamics that a repeller (i.e. the complementary of the basin of an attracting set) varies upper semi-continuously with the parameter. And, by Theorem 2.3.2, in the Desboves family the small Julia set is a repeller (the complementary of the basin of ρ_0). Hence, $\mathscr{J}_2(\lambda)$ depends continuously on λ in the Desboves family.

The proof of the second point is also simple. By Theorem 2.2.1, it is sufficient to prove that Misiurewicz parameters are dense in $\mathbb{C}^*$. The first observation is that, since each map f_λ in the elementary Desboves family preserves the same pencil of lines, the critical set C_{f_λ} of f_λ is not irreducible. It admits a decomposition

$$C_{f_\lambda} = C_\infty(\lambda) \cup C_\sigma(\lambda),$$

where $C_\infty(\lambda)$ is fibered by lines of the pencil and $C_\sigma(\lambda)$ is generically transverse to the pencil. Since the action of f_λ on the pencil is independent of λ, the sets $C_\infty(\lambda)$ and $f_\lambda^n(C_\infty(\lambda))$ are also independent of λ. Hence, the bifurcations in the family cannot come from this fibered part of the critical set. On the other hand, the points of intersection of $C_\sigma(\lambda)$ with $Y := \{y = 0\}$ depend on λ and the preimages of the repelling fixed point $r = [1 : 0 : -1]$ by $(f_\lambda)_{|Y}$ are dense in Y and independent of λ. Hence, for each $\lambda_0 \in \mathbb{C}^*$ there exists λ_1 arbitrarily close to λ_0 and $n_1 \geq 1$ such that $r \in f_{\lambda_1}^{n_1}(C_\sigma(\lambda_1))$. To prove that λ_1 is a Misiurewicz parameter, it remains to show that r is not in $f_\lambda^{n_1}(C_\sigma(\lambda))$ for all $\lambda \in \mathbb{C}^*$. In general, such question can be very difficult without an ad-hoc argument. In [BT17], we use the degenerescence of the maps f_λ when λ goes to 0 to prove that such an intersection cannot be persistent. □

We conclude this section with some additional interesting properties of the elementary Desboves maps.

Remark 2.3.3. *1) It is easy to check that the point* $[0:0:1]$ *is fixed by* f_λ *for every* $\lambda \in \mathbb{C}^*$. *It has one repelling eigenvalue and the other one is equal to* $1+\lambda$. *Hence, if* $1+\lambda$ *is of modulus* 1 *and satisfies the Brjuno arithmetic condition then there exists a one-dimensional Siegel disk passing through* $[0:0:1]$. *By Theorem 2.3.2, this disk has to be contained in* $\mathscr{J}_2(\lambda)$. *This was our original motivation to consider this family of maps since by a result in [BBD18], if the small Julia set moves continuously in a family with bifurcations it must exist parameters where the small Julia set contains a Siegel disk.*

2) The elliptic curve $\mathcal{F} = \{x^3+y^3+z^3=0\}$ *is invariant by* f_λ *and* $(f_\lambda)_{|\mathcal{F}}$ *is an expanding map. Moreover, the Lebesgue measure* ν *on* $\mathcal{F}$ *is ergodic so it must admit a positive Lyapunov exponent. If the second Lyapunov exponent is negative (and numerical computations in [BDM07] suggest this is the case for* $\lambda \in \mathbb{C}^*$ *close enough to* 0*) then the union of the stable manifolds of* ν *has positive Lebesgue measure in* $\mathbb{P}^2$. *These stable manifolds are contained in* $\mathscr{J}_2(\lambda)$ *by Theorem 2.3.2 and thus, for these parameters* λ, *the small Julia set* $\mathscr{J}_2(\lambda)$ *has positive Lebesgue measure.*

3) With this parametrization, f_λ *degenerates to a rational self-map of* $\mathbb{P}^2$ *when* λ *goes to* 0 *(see [BDM07, Remark 6.9] for more properties of* f_0*). If we use a different parametrization* $F_\lambda := (\phi_\lambda)^{-1} \circ f_{\lambda^3} \circ \phi_\lambda$ *where* $\phi_\lambda[x:y:z] = [\lambda x : y : \lambda z]$ *then the family* $(F_\lambda)_{\lambda\in\mathbb{C}^*}$ *extends at* 0 *and* F_0 *is an endomorphism of* $\mathbb{P}^2$ *with some special properties. For instance,* F_0 *still satisfies Theorem 2.3.2 and every non-fixed periodic point in the invariant line* $Y=\{y=0\}$ *has one repelling eigenvalue and the other one is equal to* 1, *i.e.* $\mathscr{J}_2(F_0)$ *contains infinitely many parabolic-repelling periodic points (and their basins).*

4) In the opposite direction, when λ *converges to infinity then Favre proved in [Fav16] that the sum* $L(\lambda)$ *of the Lyapunov exponents of the equilibrium measure of* f_λ *satisfies*

$$L(\lambda) = L_{\mathrm{na}} \log|\lambda| + o(\log|\lambda|),$$

where L_{na} *can be interpreted as the sum of the Lyapunov exponents of a non-Archimedean dynamical system. For the elementary Desboves family, one can prove that* $L_{\mathrm{na}} = 1/4$.

2.4 Blenders

The purpose of a blender is to "blend" in a robust way different parts of the dynamics. For instance, it is often used to obtain robust intersections between the stable set $W^s_{\Lambda_1}$ and the unstable set $W^u_{\Lambda_2}$ of two hyperbolic sets Λ_1, Λ_2, while the sum of the dimension of the stable bundle of Λ_1 with the one of the unstable bundle of Λ_2 is strictly smaller than the dimension of the ambiant space. There exist several mechanisms giving such results but blenders have the advantage to be simple and flexible. On feature of this is the result of Bonatti-Díaz [BD08] which shows that blenders always appear in the $\mathcal{C}^1$-unfolding of a heterodimensional cycle and make this cycle robust (under some mild assumptions).

In a holomorphic setting, it seems difficult to obtain such a universal result. However, my main result in this subject is that blenders always appear near bifurcations of product maps of $\mathbb{C}^2$ and give rise to open sets of bifurcations in $\mathscr{H}_d(\mathbb{P}^2)$ (see [Taf17]). Notice that, since bifurcations in the sense of [BBD18] are consequences of interplays between the small Julia set and the postcritical set, blenders will be used in what follows to obtained robust intersection between these two sets.

Blenders have also been used by Biebler in [Bie16] in order to obtain persistent homoclinic tangencies for polynomial automorphisms of $\mathbb{C}^3$ of small degree. And, in his work on bifurcations of Lattès maps [Bie19], he uses a construction which has the flavor of "para-blenders" introduced by Berger [Ber16].

2.4.1 Constructions of blenders and a toy model

Unlike the Newhouse phenomenon, the construction of blenders relies on elementary arguments (mainly a covering property combined with the contraction of a cone field) and, as we will see, simple toy models exist.

Originally, the notion of blenders was introduced (see [BD96]) for diffeomorphisms on smooth manifolds of dimension larger than or equal to 3 since it needs at least 3 distinct directions: one strong stable direction, one strong unstable direction and one weak stable or unstable direction. In our non-invertible setting, the construction can be started in dimension 2 since the non-injectivity can be considered as an additional stable direction which is especially strong: the preimages of a point x converge in finite time to x. For simplicity, in what follows we only consider the 2-dimensional case. Hence, we will obtain two types of blenders. If the center direction is unstable then the blender will be of *repelling type* and if the center direction is stable then it will be of *saddle type*.

All the maps that we will use are perturbations of product maps of the form

$$(z,w) \mapsto (p(z), q(w)).$$

Hence, there are two natural directions. The *horizontal direction* is the one parallel to $\{w = c\}$ and the *vertical direction* is the one parallel to $\{z = c\}$. The vertical direction will always be close to our strong unstable direction.

Roughly speaking, the idea behind blenders of repelling type for a skew product $f(z,w) = (p(z,w), q(w))$ of $\mathbb{C}^2$ is the following. Let $H_1, \dots, H_N$ and $V_1, \dots, V_N$ be $2N$ open sets in $\mathbb{C}$ and define $H := \cup_{s=1}^N H_s$, $V := \cup_{s=1}^N V_s$ and $Z := \cup_{s=1}^N H_s \times V_s$. The set Z contains a blender of repelling type if for each $1 \leq s \leq N$

- q is (strongly) expanding on V_s and $\overline{V} \subset q(V_s)$,
- p is (weakly) expanding in the horizontal direction on $H_s \times V_s$ and $\overline{p(H_s \times V_s)} \subset H$.

Even if f is repelling on Z, its geometric behavior and its action on the tangent space (one direction is much more expanded that the other) both mimic those of a saddle set. And actually, the "local stable set" (given by $\Lambda := \cap_{n\geq 0} f^{-n}(\overline{Z})$ and which we refer to as the blender) of the maximal invariant set of f in Z behaves as a one dimensional stable manifold: any vertical graph passing through Z has to intersect it (see Example 2.4.1 below for the arguments in a simple setting). Moreover, these properties are stable under small perturbations. These are the main two properties of a blender of repelling type: intersection with a family of graphs and robustness.

Example 2.4.1. *Here is an example of a repelling blender coming from [BCDW16]. In that paper, which focus on diffeomorphisms, it is called a* proto-blender *since the map is not injective. By replacing this non-injectivity by a standard strong stable direction, one obtains a blender for a diffeomorphism in dimension* 3.

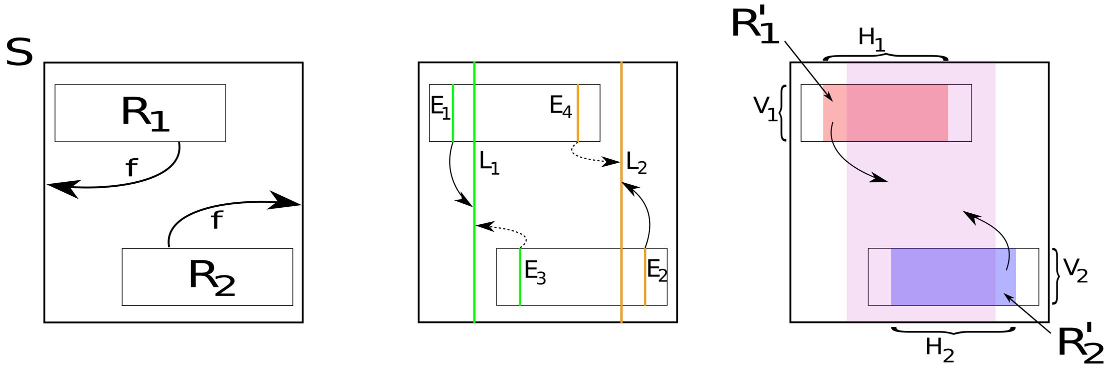

Figure 2.1 – A repelling blender in $\mathbb{R}^2$.

Consider the square $S = [0,1]^2$ in $\mathbb{R}^2$ and two rectangles $R_1, R_2 \subset S$ place as in Figure 2.1. Let f be the map defined on $R_1 \cup R_2$ such that, for $s \in \{1,2\}$, $f_{|R_s}$ is affine, preserves the vertical and horizontal directions and satisfies $f(R_s) = S$. The important point about the position of the two rectangles R_1 and R_2 is that they admit sub-rectangles R'_1 and R'_2 such that

- *the image by f of the leftmost side E_1 of R'_1 is a vertical segment L_1 which intersects R'_1,*
- *the image by f of the rightmost side E_2 of R'_2 is a vertical segment L_2 which intersects R'_2,*
- *the other preimage E_3 by f of L_1 is to the left of the other preimage E_4 of L_2.*

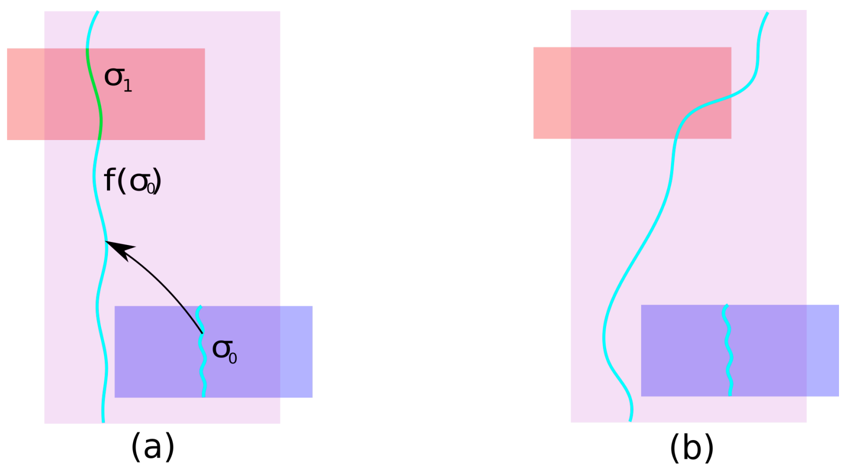

Figure 2.2 – **(a)** The image of a vertical graph σ_0 in R'_2 must intersect R'_1 or R'_2 in a vertical graph σ_1. **(b)** What could happen without the cone field condition.

From this, it is easy to check that, if we define H_1, H_2, V_1, V_2 *such that* $R'_1 = H_1 \times V_1$ *and* $R'_2 = H_2 \times V_2$ *then the map* f *has a blender of repelling type in* $Z := R'_1 \cup R'_2$ *as defined above.*

The set $\Lambda := \cap_{n \geq 0} f^{-n}(S)$ *is a Cantor set in* $\mathbb{R}^2$ *whose projection on the first coordinates has non-empty interior. Actually, a must stronger property holds: if* σ_0 *is a vertical path (i.e. whose tangent vectors are sufficiently vertical) passing through* R'_2 *(or* R'_1*) then its image must intersect* R'_1 *or* R'_2 *in a vertical graph* σ_1 *(see Figure 2.2). The same holds for the image of* σ_1 *and by induction, this gives a vertical graph* σ_n *contained in* $f^n(\sigma_0)$*. Hence, the point* $\cap_{n \geq 0} f^{-n}(\sigma_n)$ *is in the intersection of* σ_0 *with* Λ*, i.e. every vertical graph passing through* R'_1 *or* R'_2 *has to intersect* Λ*. This is the blender property in the repelling case. Moreover, this property also holds for small* $\mathcal{C}^1$ *perturbations of* f*.*

Roughly speaking, a blender of saddle type is a blender of repelling type for "f^{-1}" by taking into account that the non-injectivity can be seen as a strong stable direction. Notice that, since the contraction in this direction is particularly strong, no cone field condition is required and a saddle blender is simply a hyperbolic saddle set Λ whose local unstable set $W^u_{\Lambda,\mathrm{loc}}$ has non-empty interior.

Here is a simple way to obtain blenders (of repelling type or of saddle type) near maps of the form $(z, w) \mapsto (z, w^{4l})$ with $l \geq 1$ large. For these examples, the integer N above is equal to 3.

Example 2.4.2 (Toy models of blenders in complex dynamics)**.** *Let* $\epsilon > 0$ *be small and let* $\alpha > 0$ *be much smaller than* ϵ*. Let* H *be the (open) triangle in* $\mathbb{C}$ *with vertices* 1*,* j *and* j^2 *where* $j := e^{2i\pi/3}$*. For* $s \in \{1, 2, 3\}$*, let denote by* H_s *the image of* H *by the homothety of center* j^s *and ratio* $1 - \epsilon$*. It is easy to check, since* $\epsilon > 0$ *is small, that* $H = \cup_{s=1}^3 H_s$*. Moreover, the affine maps*

$$\varphi_s(z) = (1 + \alpha)(z - \epsilon j^s) \quad \textit{and} \quad \psi_s(z) = (1 - \alpha)(z + \epsilon j^s)$$

satisfy

$$\overline{\varphi_s(H_s)} \subset H \quad \textit{and} \quad \overline{H} \subset \cup_{s=1}^3 \psi_s(H_s).$$

(see Figure 2.3 and Figure 2.4). These inclusions persist under small perturbations. Hence, if for each $s \in \{1, 2, 3\}$ V_s *is a very small neighborhood of* j^s *and if* $l \geq 1$ *is large enough in order to have*

$$\overline{V_1 \cup V_2 \cup V_3} \subset q(V_s),$$

where $q(w) := w^{4l}$*, then the maps*

$$f(z, w) = ((1 + \alpha)(z - \epsilon w), w^{4l}) \quad \textit{and} \quad g(z, w) = ((1 - \alpha)(z + \epsilon w), w^{4l})$$

have a blender of repelling type and a blender of saddle type respectively contained in

$$Z := \cup_{s=1}^3 H_s \times V_s.$$

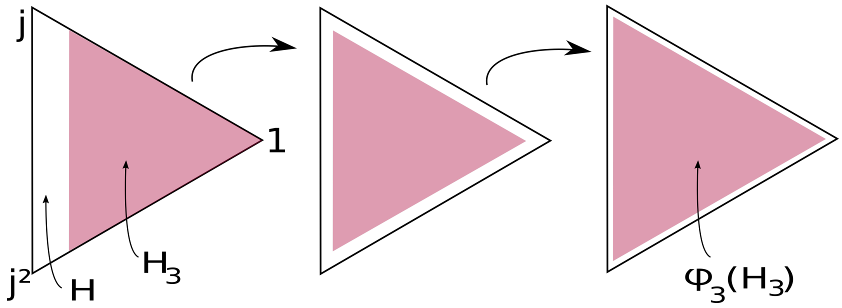

Figure 2.3 – The horizontal dynamics in a repelling blender (translation then small expansion).

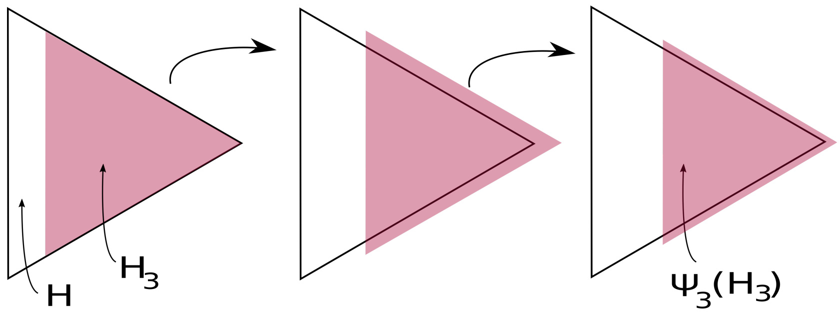

Figure 2.4 – The horizontal dynamics in a saddle blender (translation then small contraction).

2.4.2 Existence near product bifurcations and consequences

The main result of [Taf17] is to show that constructions similar to Example 2.4.2 can be achieved near $(z,w)\mapsto(p(z),q(w))$, where p,q are two degree d polynomials and p is in the bifurcation locus.

Theorem 2.4.3. *Let $d\geq 2$ and let $\mathscr{P}_d$ be the family of all one variable degree d polynomials. If p and q are two elements of $\mathscr{P}_d$ such that p is in the bifurcation locus of $\mathscr{P}_d$ then the map $(p,q)\in\mathscr{H}_d(\mathbb{P}^2)$ can be approximated both by polynomial skew products of the form $(z,w)\mapsto(\widetilde{p}(z,w),q(w))$ having an iterate with a blender of repelling type and by others having an iterate with a blender of saddle type.*

It turns out to be quite easy to replace w^{4l} in Example 2.4.2 by a sufficiently large iterate of q. The role of $1,j,j^2$ will then be fulfilled by any triplet of distinct repelling periodic points of q (see [Taf17, Lemma 3.1]). On the other hand, since p is in the bifurcation locus of $\mathscr{P}_d$, it can be approximated by a polynomial $\widehat{p}$ with a parabolic periodic point which can be assume, after a change of coordinates, to be at 0. Hence, some iterate of $\widehat{p}$ satisfies $(\widehat{p}^m)'(0)=1$, i.e. in small neighborhoods of 0 $\widehat{p}^m$ looks like $z\mapsto z$. Therefore, it seems reasonable to expect that small perturbations of $(z,w)\mapsto(\widehat{p}^{ml}(z),q^{ml}(w))$ possess blenders. All the difficulty in Theorem 2.4.3 is to obtain these perturbations as the (ml)-th iterate of perturbations of $(z,w)\mapsto(\widehat{p}(z),q(w))$ (see [Taf17, Theorem 4.1]).

Observe that blenders by themselves do not create bifurcations as they are hyperbolic sets. Their existence has important consequences when they "blend" together different parts of the dynamics. To insure that this happens, we need additional information on the dynamics. The construction above is explicit and as a corollary (in the repelling case), we obtain a positive answer to a conjecture of Dujardin [Duj17, Theorem 5.6].

Corollary 2.4.4. *The bifurcation locus of the family $\mathscr{P}_d \times \mathscr{P}_d$ is contained in the closure of the interior of the bifurcation locus in $\mathscr{H}_d(\mathbb{P}^2)$.*

On the other hand, the unstable set of a blender of saddle type has non-empty interior. Hence, we can use it to obtain large attracting sets.

Corollary 2.4.5. *The bifurcation locus of the family $\mathscr{P}_d \times \mathscr{P}_d$ is contained in the closure of the interior of the set of maps in $\mathscr{H}_d(\mathbb{P}^2)$ possessing a proper attracting set with non-empty interior.*

To the best of my knowledge, no previous example of proper attracting sets with non-empty interior was known. The result above says that they are abundant. However, as these attracting sets come from polynomial skew products of $\mathbb{C}^2$ they possess an attracting cycle near the line at infinity and thus they cannot be attractors. Nevertheless, using blenders of saddle type, it is quite easy to obtain attractors with non-empty interior in $\mathbb{P}^2$ (see Theorem 1.6.9 and [Taf17, Theorem 1.5]).

Blenders have been introduced in [BD96] in order to obtain robustly transitive diffeomorphisms which are isotopic to the identity on some 3-manifolds M. A key point (already use in previous examples) is to create a periodic saddle point p whose stable manifold and unstable manifold are both dense in M, in a robust way. In dimension 3, one of the invariant manifolds (say W^s_p) is 2-dimensional and the other one (W^u_p) is 1-dimensional. In this situation, the condition $\overline{W^s_p} = M$ is much easier to satisfy than $\overline{W^u_p} = M$. The idea in [BD96] is to consider two saddle points, p and q, with different indices ($\dim(W^s_p) = 2$, $\dim(W^s_q) = 1$), which satisfy (in a robust way)

$$\overline{W^s_p} = M = \overline{W^u_q},$$

and such that

- q is homoclinically related to a blender Λ of stable index 1,
- the unstable manifold of p intersects robustly the stable manifolds of Λ, thanks to the blender property.

All this ensure that $\overline{W^s_q} = \overline{W^s_p}$, i.e. both the stable and unstable manifolds of q are dense in M.

It is natural to wonder whether such construction can be archived in complex dynamics. Notice that for endomorphisms of $\mathbb{P}^k$, every repelling periodic point r in the small Julia set satisfies

$$\overline{W^u_r} = \mathbb{P}^k.$$

Hence, using the same idea as above, on can hope to construct a saddle point p in $\mathbb{P}^2$ such that

$$\overline{W^u_p} = \mathbb{P}^2,$$

in a robust way. Actually, this is the case for the examples given in the theorem below. This could be a first step in order to obtain robustly transitive endomorphisms of $\mathbb{P}^2$. However, it seems difficult to construct a saddle point p such that $\overline{W^s_q} = \mathbb{P}^2$.

Theorem 2.4.6. *There exist $d \geq 2$ and an open set $\Omega \subset \mathscr{H}_d(\mathbb{P}^2)$ which contains skew products such that $\Omega \subset \mathrm{Bif}(\mathscr{H}_d(\mathbb{P}^2))$ and each f in Ω possesses a hyperbolic set of saddle type Λ with positive entropy whose unstable set W^u_Λ is a Zariski open set of $\mathbb{P}^2$ and*

- *for each $\widehat{x}$ in the natural extension of Λ, the unstable manifold $W^u_{\widehat{x}}$ is dense in $\mathbb{P}^2$,*

- *for each* $x \in \Lambda$*, the stable manifold* W_x^s *is contained in the small Julia set* $\mathscr{J}_2(f)$*.*

In particular, we have $\Lambda \subset \mathscr{J}_2(f)$ *and the postcritical set of* f *is dense in* $\mathbb{P}^2$*.*

The particularity of this family is to possess two blenders. The first one (of saddle type) is the hyperbolic set Λ in the statement. The second, Λ_r, is of repelling type and is contained in the small Julia set. Moreover, these two blenders form a “heterodimensional cycle”. In this situation, this simply means that the unstable manifolds of Λ intersect Λ_r. Observe that all the properties in Theorem 2.4.6 are all consequences of this intersection.

Notice also that the small Julia set is by definition the support of the equilibrium measure of f which is of repelling nature (all its Lyapunov exponents are positive [BD99]). The above statement in the case of repelling hyperbolic sets is classical but this result is the first example, to the best of my knowledge, of a saddle hyperbolic set which is robustly contained in the small Julia set. It also provides the first example of an endomorphism of $\mathbb{P}^2$ with a saddle point whose unstable manifold is dense (and moreover in a robust way in $\mathscr{H}_d(\mathbb{P}^2)$).

Chapter 3

Dynamics of fibered endomorphisms

These last years, many new interesting dynamical phenomena in $\mathbb{P}^2$ have been exhibited using polynomial skew products (see e.g. [Duj16], [ABD$^+$16], [Duj17], [Taf17]). Before these works, a lot of information has been obtained on this type of maps by Jonsson in [Jon99], helping the future developments. Skew products are particular examples of endomorphisms of $\mathbb{P}^2$ preserving a fibration and, as we have seen in Section 2.3 (see also [BT17]), other families of endomorphisms with this property can have interesting features. Hence, it is natural to extend the results of [Jon99] to a broader framework. This is precisely the motivation of [DT18b] with Dupont. Most of the proofs in this paper are elementary but its results may be seen as a toolbox which can be useful to study specific examples.

3.1 Fibered endomorphisms of $\mathbb{P}^k$

An endomorphism f of $\mathbb{P}^k$ of degree $d \geq 2$ preserves a fibration if there exist a compact analytic space X of dimension r $(1 \leq r \leq k-1)$, a meromorphic dominant map $\pi\colon \mathbb{P}^k \to X$ and a meromorphic self-map θ of X such that

$$\pi \circ f = \theta \circ \pi.$$

When $k = 2$, the work [FP11] of Favre-Pereira implies that X has to be equal to $\mathbb{P}^1$ and the map θ is holomorphic. In higher dimension, very little is known on the fibrations preserved by endomorphisms. In particular, are there interesting examples where

- X is singular or,
- θ is not holomorphic or,
- the dimension of the indeterminacy set $I(\pi)$ of π is larger than the dimension of the generic fiber (i.e. $k - \dim(X)$) ?

By "interesting", we want to exclude triplets $(\widetilde{\pi}, \widetilde{X}, \widetilde{\theta})$ obtained from a standard example (say $X = \mathbb{P}^2$, $\pi\colon \mathbb{P}^k \to \mathbb{P}^2$ is the linear fibration) by

$$\widetilde{\pi} = \phi \circ \pi, \quad \widetilde{\theta} = \phi \circ \theta \circ \phi^{-1}$$

where $\phi\colon X \to \widetilde{X}$ is a bimeromorphic map. Getting a better understanding on the possible fibrations preserved by endomorphisms of $\mathbb{P}^k$ seems to be an interesting algebraic problem, which is out of my scope of competence. In [DT18b], we work in the following setting, which cover the few known examples:

- $X = \mathbb{P}^r$ with $1 \leq r < k$,
- θ is holomorphic,
- the indeterminacy set $I(\pi)$ of π is disjoint from the Julia set $\mathscr{J}_q(f)$ of order $q := k-r$ of f.

This last assumption implies in particular that $\dim(I(\pi)) \leq q-1$. Observe that the assumption $X = \mathbb{P}^k$ is not restrictive as long as X in smooth and $\dim(I(\pi)) \leq q-1$. Actually, in this situation the restriction of π to a generic linear subspace of dimension r in $\mathbb{P}^k$ gives a surjective holomorphic map from $\mathbb{P}^r$ to X. Then by results in [DHP08, Section 2 & 3], X is projective and then by [Laz84], X is isomorphic to $\mathbb{P}^r$.

As we assume that θ is holomorphic, both f and θ possess Green currents, T_f^i and T_θ^j respectively for $i \in \{1, \dots k\}$ and $j \in \{1, \dots, r\}$. The equilibrium measures of f and θ are $\mu_f := T_f^k$ and $\mu_\theta := T_\theta^r$. Since f and θ are semi-conjugated by π, a natural question is whether there exists a relation between T_f^i and the pull back of T_θ by π. The main result in [DT18b] gives such a relationship if $i > q$. More precisely, if we define

$$S := \frac{\pi^* T_\theta}{\|\pi^* T_\theta\|}$$

then we have the following result.

Theorem 3.1.1. *Let $f\colon \mathbb{P}^k \to \mathbb{P}^k$ and $\theta\colon \mathbb{P}^r \to \mathbb{P}^r$ be two endomorphisms of degree $d \geq 2$. Assume there exists a dominant rational map $\pi\colon \mathbb{P}^k \dashrightarrow \mathbb{P}^r$ whose indeterminacy set $I(\pi)$ is disjoint from $\mathscr{J}_q(f)$ and such that $\theta \circ \pi = \pi \circ f$. Then for $j \in \{1, \dots, r\}$, the current S^j is well-defined, satisfies $S^j \neq T_f^j$ and $T_f^{q+j} = T_f^q \wedge S^j$. In particular, $\mu_f = T_f^q \wedge S^r$ and $\pi_* \mu_f = \mu_\theta$.*

The proof is simple, it only relies on the properties of the currents T_f and T_θ and is coordinate free. Here is a brief sketch of it.

Sketch of proof. Since $I(\pi) \cap \mathscr{J}_q(f) = \varnothing$, there exists a neighborhood U of $I(\pi)$ such that $U \cap \mathscr{J}_q(f) = \varnothing$. If $\omega_{\mathbb{P}^r}$ and $\omega_{\mathbb{P}^k}$ are the Fubini-Study forms on $\mathbb{P}^r$ and $\mathbb{P}^k$ respectively then $R := \pi^*(\omega_{\mathbb{P}^r})$ is smooth on $\mathbb{P}^k \setminus U$ and there exists $C > 0$ such that $R \leq C\omega_{\mathbb{P}^k}$ on $\mathbb{P}^k \setminus U$. Hence, it follows from $U \cap \mathscr{J}_q(f) = \varnothing$ that for $j \in \{1, \dots, r\}$

$$T_f^q \wedge R^j \leq C^j (T_f^q \wedge \omega_{\mathbb{P}^k}^j).$$

By applying $d^{-n(q+j)}(f^n)^*$ on both sides this gives

$$T_f^q \wedge \left(\frac{1}{d^{nj}} (f^n)^* R^j \right) \leq C^j \left(T_f^q \wedge \left(\frac{1}{d^{nj}} (f^n)^* \omega_{\mathbb{P}^k}^j \right) \right). \tag{3.1}$$

Standard equidistribution results imply that the right-hand side converges to $C^j T_f^{q+j}$. We also deduce from the relation $\pi \circ f = \theta \circ \pi$ that $d^{-nj}(f^n)^* R^j = \pi^*(d^{-nj}(\theta^n)^* \omega_{\mathbb{P}^r}^j)$ which converges to $\pi^*(T_\theta^j)$, i.e. a constant times S^j. All this implies that there exists a constant $\widetilde{C} > 0$ such that

$$T_f^q \wedge S^j \leq \widetilde{C} T_f^{q+j}.$$

Moreover, both these currents are totally invariant and T_f^{q+j} is an extremal elements in the cone of such currents (see [DS09, Theorem 5.4.1]). Therefore, $T_f^q \wedge S^j$ is proportional to T_f^{q+j} and thus they must be equal as they have the same mass. □

Using the classifications obtained in [DJ08] and [FP11], one can check easily that the assumption $\mathscr{J}_q(f) \cap I(\pi) = \varnothing$ is always satisfied when $k = 2$, in which case $q = 1$. This is also the case when π is the standard linear fibration defined by $\pi[y : z] = [y]$ with $y := (y_0, \ldots, y_r) \in \mathbb{C}^{r+1}$ and $z := (z_0, \ldots, z_{q-1}) \in \mathbb{C}^q$. In general, I know no example where this assumption does not hold.

Remark 3.1.2. *Using others results of Dabija-Jonsson [DJ10] and Favre-Pereira [FP15], one can see that the proof above also applies to endomorphisms of $\mathbb{P}^2$ preserving an algebraic web. But in some examples, the counterpart of S is equal to T_f.*

The main point in Theorem 3.1.1 is the formula $\mu_f = T_f^q \wedge S^r$ which can be seen as a generalization of the decomposition of μ_f obtained by Jonsson [Jon99] for polynomial skew products of $\mathbb{C}^2$. Indeed, for μ_θ-almost every $a \in \mathbb{P}^r$ the fiber $L_a := \overline{\pi^{-1}(a)}$ has dimension q and we can define the probability measure

$$\mu_a := \frac{T_f^q \wedge [L_a]}{\|\pi^* T_\theta\|^r}.$$

Corollary 3.1.3. *Let $\phi\colon \mathbb{P}^k \to \mathbb{R}$ be a continuous function. Under the assumptions of Theorem 3.1.1 we have*

$$\int_{\mathbb{P}^k} \phi(x) d\mu_f(x) = \int_{\mathbb{P}^r} \left(\int_{L_a} \phi(x) d\mu_a(x) \right) d\mu_\theta(a).$$

In particular, the small Julia set $\mathscr{J}_k(f)$ satisfies

$$\mathscr{J}_k(f) = \overline{\cup_{a \in \mathscr{J}_r(\theta)} \mathrm{supp}(\mu_a)}.$$

In particular, if $k = 2$ then the small Julia set is exactly equal to the closure of the set of repelling periodic points (see [Dab00] for another proof of this fact).

The formula $\mu_f = T_f^q \wedge S^r$ has other consequences. For instance, if μ_θ is absolutely continuous with respect to Lebesgue measure (i.e. θ is a Lattès mapping of $\mathbb{P}^r$, see [BD05]) then μ_f is absolutely continuous with respect to the trace measure $\sigma_{T_f^q} := T_f^q \wedge \omega_{\mathbb{P}^k}^r$.

Corollary 3.1.4. *Under the assumptions of Theorem 3.1.1, if $\mu_\theta << \omega_{\mathbb{P}^r}^r$ then $\mu_f << \sigma_{T_f^q}$.*

That applies to the elementary Desboves mappings of $\mathbb{P}^2$ considered in Section 2.3 since they induce a Lattès mapping on a pencil of lines. Let us note that when $k = 2$, the property $\mu_f << \sigma_{T_f}$ implies that the smallest exponent of μ_f is minimal, equal to $\frac{1}{2} \log d$, see [Duj12, Theorem 3.6]. In particular the Lyapunov exponents of Desboves mappings are $\lambda_1 > \lambda_2 = \frac{1}{2} \log d$, with $d = 4$. The following Theorem generalizes that semi-extremal property to fibered endomorphisms satisfying $\mu_\theta << \omega_{\mathbb{P}^r}^r$. It is a consequence of $\pi_* \mu_f = \mu_\theta$ and holds for more general smooth dynamical systems.

Theorem 3.1.5. *Let f, π and θ be as in Theorem 3.1.1. If Λ is a Lyapunov exponent of multiplicity m for μ_θ then Λ is a Lyapunov exponent of multiplicity at least m of μ_f.*

In particular, the sum Λ_f (resp. Λ_θ) of all the Lyapunov exponents of f (resp. θ) satisfies

$$\Lambda_f = \Lambda_\theta + \Lambda_\sigma,$$

where Λ_σ is the sum of Lyapunov exponents in the direction of the fibers of π. If $(f_\lambda)_{\lambda \in M}$ is a family of endomorphisms of $\mathbb{P}^k$ which all preserve the same fibration (i.e. $\pi \circ f_\lambda = \theta_\lambda \circ \pi$ where $(\theta_\lambda)_{\lambda \in M}$ is a family of endomorphisms of $\mathbb{P}^r$) then the bifurcation current

$T_{\mathrm{Bif}}(f) := dd^c\Lambda_f$ has a decomposition into the sum of two currents $T_{\mathrm{Bif}}(\theta) := dd^c\Lambda_\theta$ and $T_{\mathrm{Bif},\sigma} := dd^c\Lambda_\sigma$. A natural question is whether the current $T_{\mathrm{Bif},\sigma}$ is positive, i.e. $\lambda \mapsto \Lambda_\sigma(\lambda)$ is plurisubharmonic. Moreover, one can expect that this current can be expressed in terms of currents in the product $M \times \mathbb{P}^k$. This seems difficult to archive in full generality (one difficulty is that the pull back of the critical set of θ by π may not be include in the critical set of f) but we obtain such results in [DT18b] when π is the linear fibration (see [DT18b, Theorem 1.7]). Notice that in [AB18] Astorg-Bianchi study the bifurcations in families of polynomial skew products of $\mathbb{C}^2$ in a much deeper way.

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